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SYMMETRIC PRESENTATIONS AND RELATED TOPICS

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Mayra McGrath

March 2020

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ABSTRACT

In this thesis, we have investigated several permutation and monomial progenitors for finite images. We have found original symmetric presentations for several important non-abelian simple groups, including linear groups, unitary groups, alternating groups, and sporadic simple groups.

We have found a number of finite images, including $2^4 : 3^3$, $2 \times S_7$, $2^4 : (2^2 : 3^2)$, $L(2,19)$, $L(2,41)$, $PSL(2,11) \times 2$, $L(2,8)$, $L(2,19)$, $M_{21} = L(3,4)$, M_{12} , $Aut(M_{12})$, as homomorphic images of the permutation progenitors $2^{*6} : (3^2 : 2^2)$, $2^{*6} : A_5$, $2^{*100} : (A_5 \times D_{10})$, and $2^{*12} : S_4$. We have also found $PGL(2,16) : 2 = Aut(PSL(2,16))$ and $PSL(2,16)$ as homomorphic images of the monomial progenitors $17^{*4} :_m (2^3 : \bullet 4)$ and $3^{*3} :_m A_4$, respectively.

We have performed manual double coset enumeration to construct $U(3,5):2$ over $PGL(2,9)$ as well as $U(3,5):2$ over the maximal subgroup $Aut(A_6)$ and $PGL(2,9)$, $PSL(2,11)$ over A_4 as well as over the maximal subgroup A_5 and A_4 , as well as, $PSL(2,11)$ over A_5 . In addition, we have given the isomorphism class of each image that we have discovered. Presentation for all the progenitors that we have studied along with their images is presented in tables.

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Introduction

The infinite semi-direct product, called a progenitor, $m^{*n} : N$, where m^{*n} is a free product of n cyclic groups generated by t_i , $1 \leq i \leq n$, of order m and N is a group of automorphisms of m^{*n} which permutes the n cyclic subgroups by conjugation. Therefore, for any $k \in N$, $t_i^k = t_i^l$, where $\gcd(m,l)=1$. When $m = 2$, N acts as a permutation group on the n indices of the t_i s; that is, $N \leq S_n$. The elements of a progenitor can be written, not necessarily uniquely, as πw , where $\pi \in N$ and w is a word in the t_i s.

Then any relation that a progenitor can be factored by to give a finite image must take the form $nw = 1$. We will represent this factorization as $\frac{m^{*n}:N}{nw}$. We will demonstrate how such factor groups can be identified. We also note that frequently a progenitor factored by a single relation gives a simple group and even a sporadic simple group. Thus, progenitors pave the way for a number of topics, related to presentations and representations, that will be discussed through out the next few chapters.

In **Chapter 1**, we list definitions and theorems pertaining to progenitors. We show how to construct permutation progenitors of various free product sizes and apply Grindstaff Lemma via the computing program MAGMA to verify if we successfully built the progenitor. More precisely, we show when a progenitor is factored by given relations it allows us to find the finite homomorphic images of such progenitor. In **Chapter 2**, we discuss a different type of progenitor called the monomial progenitor. We show construction of this progenitor through a process known as the "lifting process", which allows us to build monomial matrices to obtain a new control group on which our monomial progenitor will be constructed from. In **Chapter 3**, we list the tables of the finite images discovered for each permutation and monomial progenitor, in chapters 1 and 2.

In **Chapter 4**, we solve the extension problem in order to classify the isomorphism type of transitive groups as direct product, semi-direct product, or mixed extension. In **Chapters 5-8**, we construct the double coset enumeration of a group G over a transitive group N of finite permutation and monomial progenitors. We also show the maximal double coset enumeration for two groups $U(3,5):2$ over the two groups $\text{Aut}(6)$ and $\text{PGL}(2,9)$, and monomial $\text{PSL}(2,11)$ over A_5 and A_4 . Finally, in **Chapter 9** we look at a few examples on wreath products.

Chapter 1

Permutation Progenitors

1.1 Preliminaries

Definition 1.1.1. (Permutation) *If X is a nonempty set, a **permutation** is the bijective mapping $\alpha : X \rightarrow X$. [Rot95]*

Definition 1.1.2. (Disjoint) *Two permutations $\alpha, \beta \in S_X$ are **disjoint** if every x moved by one is fixed by the other. In symbols, if $\alpha(a) \neq a$, then $\beta(a) = a$, and if $\alpha(b) = b$, then $\beta(b) \neq b$. [Rot95]*

Theorem 1.1.3. *Every permutation $\alpha \in S_n$, is either a cycle or a product of disjoint cycles. [Rot95]*

Definition 1.1.4. (Semigroup) *A **semigroup** $(G, *)$ is a nonempty set G equipped with an associative operation $*$. [Rot95]*

Definition 1.1.5. (Symmetric Group) The *symmetric group*, denoted S_n is the set of all permutations of the nonempty set $X = \{1, 2, \dots, n\}$. S_n is a group of order $n!$ on n letters. [Rot95]

Definition 1.1.6. (Group). A *group* is a semigroup G containing an element e such that

(i) $e * a = a = a * e$ for all $a \in G$

(ii) for every $a \in G$, there is an element $b \in G$ with $a * b = e = b * a$. [Rot95]

Definition 1.1.7. (Order) If G is a group, then the *order* of G , denoted $|G|$, is the number of elements in G . [Rot95]

Definition 1.1.8. (Free Group) If X is a subset of a group F , then F is a *free group* with basis X if, for every group G and every function $f : X \rightarrow G$, there exists a unique homomorphism $\varphi : F \rightarrow G$ extending f . [Rot95]

Definition 1.1.9. (Presentation) Let X be a set and let Δ be a family of words on X . A group G has generators X and relations Δ if $G \cong F/R$, where F is the free group with basis X and R is the normal subgroup of F generated by Δ . The ordered pair $(X|\Delta)$ is called a *presentation* of G . [Rot95]

Definition 1.1.10. (Progenitor) A *progenitor* is a semi-direct product of the following form: $P \cong 2^{*n} : N = \{\pi w \mid \pi \in N, \text{ and } w \text{ is a word in the } t_i\}$, where 2^{*n}

denotes a free product of n copies of a cyclic group of order 2 generated by involutions t_i for $i=1,\dots,n$; and N is a transitive permutation group of degree n which acts on the free product by permuting the involutory generators.[Curt96]

Lemma 1.1.11. (Factoring Lemma) *(Known as the Grindstaff Lemma) Factoring the progenitor $m^{*n}:N$ by (t_i, t_j) for $1 \leq i < j \leq n$ gives the group $m^n:N$. [Grind15]*

1.2 Progenitor $2^{*6} : (3^2 : 2^2)$

We wish to build the symmetric progenitor of $2^{*6} : N$. $N = \langle x, y, z \rangle \leq S_6$, where $x \sim (2, 4, 6)$, $y \sim (1, 5)(2, 4)$, $z \sim (1, 4)(2, 5)(3, 6)$, $t \sim t_1$, and $N \cong (3^2 : 2^2)$ is of order 36. The progenitor will consist of the free product, 2^{*6} , 6 groups generated by 6 t 's of order 2. The presentation of N is given by $\langle x, y, z \mid x^3, y^2, z^2, (x^{-1}y)^2, (yz)^2, x^{-1}zx^{-1}zxzxz \rangle$. We change the permutations in this presentation into letters and re-write the presentation as $\langle a, b, c \mid a^3, b^2, c^2, (a^{-1}b)^2, (bc)^2, a^{-1}ca^{-1}cacac \rangle$.

Next, we apply the Schreier System in MAGMA to correspond every word with its permutation representation. We find the stabilizer of 1 in N , denoted by N^1 . Thus, the elements of N^1 are the two permutations $(2, 4, 6)$ which corresponds to the word a (x in N), and $(3, 5)(4, 6)$ which corresponds to the word $abcac$ ($xyzxz$ in N). We must add the term t^2 since our t_i 's are of order 2.

In order to verify if the progenitor is correct we must re-write our presentation of N in terms of the stabilizer N^1 and its orbits. Thus, the orbits of N^1 are $\{1\}$, $\{3, 5\}$, and $\{2, 4, 6\}$. In MAGMA we apply the Grindstaff Lemma ([section 1.1](#)) and verify by using the following code:

```
G<x,y,z,t>:=Group<x,y,z,t| x^3,y^2,z^2,
(x^-1 * y)^2,(y * z)^2,x^-1 * z * x^-1 * z * x * z * x * z,t^2,
(t,x*y*z*x*z),(t,x),(t,t^(y*z)),(t,t^(y*z^2))>;

#G;
2304
```

2⁶*36;
2304

We note that the $G \cong 2^{*6} : (3^2 : 2^2)$. Thus, the symmetric presentation for the progenitor $2^{*6} : (3^2 : 2^2)$ is given by $\langle x, y, z, t \mid x^3, y^2, z^2, (x^{-1}y)^2, (yz)^2, x^{-1}zx^{-1}zxzxz, t^2, (t, xyzxz), (t, x) \rangle$. The symmetric progenitor is infinite but if we wish to view its finite homomorphic images we must factor this progenitor by appropriate relations.

1.2.1 First Order Relations of $2^{*6} : (3^2 : 2^2)$

To find the first order relations which are of the form $(nt_i)^k = 1$, where $n \in N$ and k is some positive power. We begin by looking at the conjugacy classes of $N \cong (3^2 : 2^2)$.

Table 1.1: Conjugacy Classes of $3^2 : 2^2$

Class	Order	Representative	Length
1	1	Id(N)	1
2	2	(1, 4)(2, 5)(3, 6)	3
3	2	(1, 2)(3, 6)(4, 5)	3
4	2	(1, 5)(2, 4)	9
5	3	(1, 3, 5)(2, 4, 6)	2
6	3	(1, 5, 3)(2, 4, 6)	2
7	3	(1, 3, 5)	4
8	6	(1, 6, 3, 2, 5, 4)	6
9	6	(1, 4, 5, 6, 3, 2)	6

Next, we find the centralizers of each conjugacy class and their respective orbits on $\{1, 2, 3, 4, 5, 6\}$. We do this as a "short cut" because, since N is of order 36 and we have 6 t'_i s, implies we would have a total of $36 \times 6 = 216$ relations for this progenitor. Thus, we use following code in MAGMA to achieve this:

```

for i in [2..#C] do i, C[i][3]; for j in [1..#N] do
if ArrayP[j] eq C[i][3] then Sch[j]; end if; end for;
Orbits(Centralizer(N,C[i][3])); end for;
2 (1, 4)(2, 5)(3, 6)
c
[
  GSet{@ 1, 5, 4, 3, 2, 6 @}
]
3 (1, 2)(3, 6)(4, 5)

```

```

b * c
[
  GSet{@ 1, 5, 2, 3, 4, 6 @}
]
4 (1, 5)(2, 4)
b
[
  GSet{@ 3, 6 @},
  GSet{@ 1, 5, 4, 2 @}
]
5 (1, 3, 5)(2, 4, 6)
(a * c)^2
[
  GSet{@ 1, 3, 2, 5, 4, 6 @}
]
6 (1, 5, 3)(2, 4, 6)
a * c * a^-1 * c
[
  GSet{@ 1, 5, 4, 3, 6, 2 @}
]
7 (1, 3, 5)
c * a * c
[
  GSet{@ 1, 3, 5 @},
  GSet{@ 2, 4, 6 @}
]
8 (1, 6, 3, 2, 5, 4)
c * a
[
  GSet{@ 1, 6, 3, 2, 5, 4 @}
]
9 (1, 4, 5, 6, 3, 2)
b * c * a
[
  GSet{@ 1, 4, 5, 6, 3, 2 @}
]

```

We change the representatives of each conjugacy class into words. For example, from the code above, conjugacy class 2 has representative $(1, 4)(2, 5)(3, 6)$ and in words this permutation corresponds to "c" (or "z" in N). Furthermore, this code tells us class 2 has one orbit which is $\{1, 5, 4, 3, 2, 6\}$. This orbit is represented as t_1 . Similarly, the class 4 representative $(1, 5)(2, 4)$ corresponds to "y", and has two orbits $\{3, 6\}$ which is represented as t_3 , such that $t^{zzz} = t_3$ (see below), and orbit $\{1, 5, 4, 2\}$ represented as t_1 .

As, explained previously the t'_i s corresponding to the orbits are determined by the following MAGMA code.

```
for j in [1..6] do for i in [1..#N] do if 1^ArrayP[i] eq j then Sch[i];
break; end if; end for; end for;
```

```
Id(NN)
b * c
c * a * c
c
b
c * a
```

The words are transcribed into the following permutations in N as:

```
yy*zz;
(1, 2)(3, 6)(4, 5)
```

```
zz*xx*zz;
(1, 3, 5)
```

```
zz;
(1, 4)(2, 5)(3, 6)
```

```
yy;
(1, 5)(2, 4)
```

zz*xx;
 (1, 6, 3, 2, 5, 4)

Since the relations are of the form $(nt_i)^k = 1$, we let n be the conjugacy class representative. The t'_i s will be represented by the respective orbit(s) pertaining to their respective conjugacy class. Thus, the first order relations are listed below:

Table 1.2: First Order Relations of $2^{*6} : (3^2 : 2^2)$

Class	First Order Relations
2	$(zt)^a$
3	$(yzt)^b$
4	$(yt)^c, (yt^{(zzx)})^d$
5	$((xz)^2t)^e$
6	$(xzx^{-1}zt)^f$
7	$(zzxt)^g, (zzxt^{(yz)})^h$
8	$(zxt)^i$
9	$(yzxt)^j$

In order to find the finite homomorphic images for this progenitor (see **Chapter 3**) we factor by the first order relations. Thus the final factored symmetric presentation is given by $\langle x, y, z, t \mid x^3, y^2, z^2, (x^{-1}y)^2, (yz)^2, x^{-1}zx^{-1}zxzxz, t^2, (t, xyzzx), (t, x), (zt)^a, (yzt)^b, (yt)^c, (yt^{(zzx)})^d, ((xz)^2t)^e, (xzx^{-1}zt)^f, (zzxt)^g, (zzxt^{(yz)})^h, (zxt)^i, (yzxt)^j \rangle$.

1.3 Progenitor $2^{*12}:S_4$

We wish to build the symmetric progenitor $2^{*12}:S_4$. We begin with a transitive group $N \cong S_4$, of order 24. $N \langle x, y \rangle \leq S_{12}$, where $x \sim (1, 2)(3, 5)(4, 6)(7, 9)(8, 10)$, $y \sim (1, 3, 6, 12)(2, 4, 7, 10)(5, 8, 11, 9)$, and $t \sim t_1$. The presentation of N is given by $\langle x, y \mid x^2, y^4, (xy^{-1})^3 \rangle$. We re-write the presentation as $\langle a, b \mid a^2, b^4, (ab^{-1})^3 \rangle$, changing all permutation into words.

We proceed as in **section 1.2** and apply the Schreier System to correspond every word with its permutation representation and identify the point stabilizer of 1 in N , denoted N^1 . $N^1 \cong \langle (2,11)(3,4)(5,10)(6,8)(7,12) \rangle = \langle x^y \rangle$. We re-write our presentation of the symmetric progenitor in terms of N^1 . Thus a presentation for the symmetric progenitor $2^{*12} : S_4$ is given by $\langle x, y, t \mid x^2, y^4, (xy^{-1})^3, t^2, (t, x^y) \rangle$.

In order to find the finite homomorphic images for this progenitor (see **Chapter 3**) we factor by the first order relations. Thus, the first order relations are: $(y^2t^x)^a, (y^2t)^b, (yxy^{-1}t)^c, (yxy^{-1}ty^2)^d, (yxy^{-1}t^x)^e, (yxy^{-1}ty^x)^f, (yxt)^g, (yxt^x)^h, (yxt^y)^i, (yxt^{xy^2})^j, (yt)^k, (yt^x)^l$, and $(yt^{yx})^m$. Thus, a final symmetric presentation for the factored symmetric progenitor is given by $\langle x, y, t \mid x^2, y^4, (xy^{-1})^3, t^2, (t, x^y), (y^2t^x)^a, (y^2t)^b, (yxy^{-1}t)^c, (yxy^{-1}ty^2)^d, (yxy^{-1}t^x)^e, (yxy^{-1}ty^x)^f, (yxt)^g, (yxt^x)^h, (yxt^y)^i, (yxt^{xy^2})^j, (yt)^k, (yt^x)^l, (yt^{yx})^m \rangle$.

1.4 Progenitor $2^{*6} : A_5$

We wish to build the symmetric progenitor $2^{*6} : A_5$, where $N = A_5 \leq S_6$ and of order 60. $N \cong \langle x, y \rangle$, where $x \sim (1,2,3,4,6)$, $y \sim (1,4)(5,6)$, and $t \sim t_1$. N is represented by the following presentation $\langle x, y \mid x^5, y^2, (yx^{-1})^3 \rangle$.

Proceeding as in **section 1.2**, we change the permutations to letters and re-write the presentation as $\langle a, b \mid a^5, b^2, (ba^{-1})^3 \rangle$, and apply the Schreier System in MAGMA in order to correspond every word in the presentation with its permutation representation. Next, we find the point stabilizer of 1 in N , denoted N^1 . $N^1 \cong \langle (3,6)(4,5), (2,4,3,6,5) \rangle = \langle xyx^{-1}, yx^2 \rangle$. Thus, the presentation of the symmetric progenitor is written in terms of N^1 as $\langle x, y, t \mid x^5, y^2, (yx^{-1})^3, t^2, (t, yx^2), (t, xyx^{-1}) \rangle$.

In order to find the finite homomorphic images for this progenitor (see **Chapter 3**) we factor by the first order relations. Thus, the first order relations are $(yt)^a, (yt^x)^b, (yt^{x^{-1}y})^c, (yxt)^d, (yxt^x)^e, (xt)^f, (xt^{x^{-1}y})^g, (x^2t^{x^{-1}y})^h, (x^2t)^i$; and the final factored symmetric presentation is given by $\langle x, y, t \mid x^5, y^2, (yx^{-1})^3, t^2, (t, yx^2), (t, xyx^{-1}), (yt)^a, (yt^x)^b, (yt^{x^{-1}y})^c, (yxt)^d, (yxt^x)^e, (xt)^f, (xt^{x^{-1}y})^g, (x^2t^{x^{-1}y})^h, (x^2t)^i \rangle$.

1.5 Progenitor $2^{*100}:(A_5 \times D_{10})$

We wish to build the symmetric progenitor $2^{*100}:(A_5 \times D_{10})$, where $N = (A_5 \times D_{10}) \leq S_{100}$ and of order 600. $N = \langle x, y \rangle$, where

$x \sim (1, 96, 19, 5, 94)(2, 71, 22, 13, 36)(3, 33, 86, 9, 34)(4, 37, 49, 56, 74)(6, 72, 51, 38, 75)(7, 54, 43, 98, 20)(8, 83, 40, 31, 45)(10, 70, 52, 61, 30)(11, 15, 91, 23, 41)(12, 35, 59, 42, 57)(14, 60, 89, 87, 66)(16, 90, 100, 62, 46)(17, 99, 53, 58, 65)(18, 32, 78, 67, 95)(21, 84, 64, 81, 97)(24, 55, 44, 93, 92)(25, 48, 50, 80, 76)(26, 73, 88, 63, 29)(27, 68, 82, 28, 85)(39, 69, 79, 47, 77),$

$y \sim (1, 61, 42, 21, 100, 9)(2, 18, 50, 84, 59, 53)(3, 88, 76, 67, 96, 86)(4, 8, 55, 15, 60, 29)(5, 98, 38, 22, 16, 79)(6, 27)(7, 32, 36, 49, 17, 93)(10, 31, 83, 74, 72, 85)(11, 68, 75, 43, 39, 89)(12, 92, 65)(13, 51, 56)(14, 62, 97, 82, 41, 26)(19, 78, 20)(23, 25, 73)(24, 57, 52, 64, 48, 91)(28, 81, 70)(30, 94, 69, 54, 44, 45)(34, 90, 71, 99, 37, 63)(35, 58)(46, 66, 47)(77, 87)(80, 95),$

and $t \sim t_1$.

N is represented by the following presentation $\langle x, y \mid x^5, y^6, (x^{-1}y^{-1})^2, x^{-1}y^{-3}xy^{-1}xy^3x^{-1}y \rangle$.

Proceeding as in **section 1.2**, we find the point stabilizer of 1 in N , denoted N^1 , is generated by $(2, 72, 99, 96, 39, 57)(3, 90, 98, 50, 91, 4)(5, 19, 85, 81, 24, 47)(6, 42, 100)(7, 40, 17, 23, 77, 25)(8, 43, 18)(9, 49, 27, 32, 61, 87)(10, 59, 79, 11, 76, 60)(12, 52, 70, 67, 20, 16)(13, 35, 51, 97, 33, 14)(15, 53, 37)(21, 93)(22, 63, 83, 55, 64, 75)(26, 95, 82, 94, 58, 45)(28, 74, 65, 38, 46, 86)(29, 71, 89)(31, 78, 88, 66, 48, 92)(34, 84, 68)(36, 73)(41, 56)(44, 80, 69)(54, 62)$; which is represented as $xy^2x^{-2}y$. Thus, the presentation for this symmetric progenitor is now re-written

in terms of N^1 and given by $\langle x, y \mid x^5, y^6, (x^{-1}y^{-1})^2, x^{-1}y^{-3}xy^{-1}xy^3x^{-1}y, t^2, (t, xy^2x^{-2}y) \rangle$. We factor this progenitor in order to obtain some finite images (see **Chapter 3**) of this progenitor.

Here, N has total of over 50 first order relations but if we add too many to the progenitor it may not run efficiently in MAGMA, so we only list a few first order relations. The first order relations are $(x^{-1}yxy^{-2}x^2t^{xy^{-1}x^2y^{-1}})^a$, $(x^{-1}yxy^{-1}xyx^{-1}yt)^b$, $(x^2yx^{-1}t^{x^{-2}})^c$, $(xy^{-1}xyt^{xy^2})^d$, $((xy^{-2})^2t)^e$, $(yx^2y^3x^{-2}y^2t)^f$, $((xy^{-1})^2t)^g$, $((xy^{-1})^4t)^h$, and $(xt)^i$. Thus, a final symmetric presentation for this progenitor is $\langle x, y, t \mid x^5, y^6, (x^{-1}y^{-1})^2, x^{-1}y^{-3}xy^{-1}xy^3x^{-1}y, t^2, (t, xy^2x^{-2}y), (x^{-1}yxy^{-2}x^2t^{xy^{-1}x^2y^{-1}})^a, (x^{-1}yxy^{-1}xyx^{-1}yt)^b, (x^2yx^{-1}t^{x^{-2}})^c, (xy^{-1}xyt^{xy^2})^d, ((xy^{-2})^2t)^e, (yx^2y^3x^{-2}y^2t)^f, ((xy^{-1})^2t)^g, ((xy^{-1})^4t)^h, \text{ and } (xt)^i \rangle$.

1.6 Progenitor $2^{*12} : PGL(2, 9)$

We wish to build the symmetric progenitor $2^{*12}:PGL(2, 9)$, where $N \cong PGL(2, 9) \leq S_{12}$ of order 720. $N = \langle x, y \rangle$, where $x \sim (2, 4, 10)(3, 5, 7)(6, 12, 8)$, $y \sim (1, 10, 9, 2, 7, 12, 11, 8, 5, 4)(3, 6)$, and $t \sim t_1$. A presentation for N is given by $\langle x, y \mid x^3, (xy^{-1})^2, y^{10}, (x^{-1}y^{-2})^4, y^3x^{-1}y^{-4}xy^3xy^{-4}x^{-1}y^2 \rangle$. As, in **section 1.2**, we find the point stabilizer N^1 , is generated by $(2, 4, 10)(3, 5, 7)(6, 12, 8) = x$, and $(2, 12, 6)(3, 11, 9)(4, 10, 8) = y^2xy^{-2}$. Thus, the presentation for the symmetric progenitor is given by $\langle x, y \mid x^3, (xy^{-1})^2, y^{10}, (x^{-1}y^{-2})^4, y^3x^{-1}y^{-4}xy^3xy^{-4}x^{-1}y^2, t^2, (t, x), (t, y^2xy^{-2}) \rangle$.

In order to find the finite homomorphic images for this progenitor (see **Chapter 3**) we factor by the first order relations. Thus, the first order relations are $(y^5t^{y^{-2}x^{-1}})^a, (y^5t)^b, ((yxy)^2t^{y^{-2}x^{-1}})^c, ((yxy)^2t)^d, (xt)^e, (xt^{yx})^f, (xt^{y^{-2}x^{-1}})^g, (xt^{y^{-2}x^{-1}y})^h, (xyxt^{y^{-2}x^{-1}})^i, (xyxt)^j, (y^2t^{y^{-2}x^{-1}})^k, (y^2t)^l, (y^{-4}t^{y^{-2}x^{-1}})^m, (y^{-4}t)^n, (yxy^2t^{y^{-1}})^o, (yxy^2t)^p, (xyxy^3xy^{-3}t^{y^{-1}})^q, (xyxy^3xy^{-3}t)^r, (yt^{y^{-2}x^{-1}})^s, (yt)^u, (y^3t^{y^{-2}x^{-1}})^v$, and $(y^3t)^w$.

We raise every first order relation to some power and inserting them into the symmetric progenitor presentation gives us the factored symmetric presentation $\langle x, y \mid x^3, (xy^{-1})^2, y^{10}, (x^{-1}y^{-2})^4, y^3x^{-1}y^{-4}xy^3xy^{-4}x^{-1}y^2, t^2, (t, x), (t, y^2xy^{-2}), (y^5t^{y^{-2}x^{-1}})^a, (y^5t)^b, ((yxy)^2t^{y^{-2}x^{-1}})^c, ((yxy)^2t)^d, (xt)^e, (xt^{yx})^f, (xt^{y^{-2}x^{-1}})^g, (xt^{y^{-2}x^{-1}y})^h, (xyxt^{y^{-2}x^{-1}})^i, (xyxt)^j, (y^2t^{y^{-2}x^{-1}})^k, (y^2t)^l, (y^{-4}t^{y^{-2}x^{-1}})^m, (y^{-4}t)^n, (yxy^2t^{y^{-1}})^o, (yxy^2t)^p, (xyxy^3xy^{-3}t^{y^{-1}})^q, (xyxy^3xy^{-3}t)^r, (yt^{y^{-2}x^{-1}})^s, (yt)^u, (y^3t^{y^{-2}x^{-1}})^v, (y^3t)^w \rangle$.

Chapter 2

Monomial Progenitors

2.1 Preliminaries

Definition 2.1.1. (Character) *Let $A(x) = (a_{ij}(x))$ be a matrix representation of G of degree m . We consider the characteristic polynomial of $A(x)$, namely*

$$\det(\lambda I - A(x)) = \begin{vmatrix} \lambda - a_{11}(x) & \lambda - a_{12}(x) & \dots & \lambda - a_{1m}(x) \\ \lambda - a_{21}(x) & \lambda - a_{22}(x) & \dots & \lambda - a_{2m}(x) \\ \dots & \dots & \dots & \dots \\ \lambda - a_{m1}(x) & \lambda - a_{m2}(x) & \dots & \lambda - a_{mm}(x) \end{vmatrix}$$

This is a polynomial of degree m in λ , with the coefficient of $-\lambda^{m-1}$ is

$$\varphi(x) = a_{11}(x) + a_{22}(x) + \dots + a_{mm}(x)$$

It is customary to call the right-hand side of this equation the trace of $A(x)$, abbreviated to $\text{tr}A(x)$, so that $\varphi(x) = \text{tr}A(x)$

*We regard $\varphi(x)$ as a function on G with values in field K , and we call it the **character** of $A(x)$. [Led77]*

Theorem 2.1.2. *The number of irreducible characters of G is equal to the number of conjugacy classes of G . [Led77]*

Definition 2.1.3. (Degree of a Character) *The sum of squares of the degrees of the distinct irreducible characters of G is equal to $|G|$. The **degree of a character** χ is $\chi(1)$. Note that a character whose degree is 1 is called a **linear character**. [Led77]*

Definition 2.1.4. (Lifting Process) *Let N be a normal subgroup of G and suppose that $A_0(Nx)$ is a representation of degree m of the group G/N . Then $A(x) = A_0(Nx)$ defines a representation of G/N lifted from G/N . If $\varphi_0(Nx)$ is a character of $A_0(Nx)$, then $\varphi(x) = \varphi_0(Nx)$ is the lifted character of $A(x)$. Also, if $u \in N$, then $A(u) = Im, \varphi(u) = m = \varphi(1)$. The lifting process preserves irreducibility. [Led77]*

Definition 2.1.5. (Induced Character) *The character of $A(x)$, which is called the **induced character** of ϕ , will be denoted by ϕ^G . Thus, $\phi^G = \text{tr}A(x) = \sum_{i=1}^n \phi(t_i x t_i^{-1})$. [Led77]*

Definition 2.1.6. (Formula for Induced Character)

$$\varphi_\alpha^G(x) = \frac{n}{h_\alpha} \sum_{\omega \in C_\alpha \cap H} = \varphi(\omega), \alpha = 1, 2, 3, \dots, m$$

2.2 Monomial Progenitor $17^{*4} :_m (2^3 : \bullet 4)$

We begin with a group G generated by $xx = (2, 6)(3, 7)$, and $yy = (1, 2, 3, 4, 5, 6, 7, 8)$. The conjugacy classes of the group G are listed below.

Conjugacy Classes of G :

$$C1 = \{\text{Id}(G)\}$$

$$C2 = \{(1, 5)(2, 6)(3, 7)(4, 8)\}$$

$$C3 = \{(2, 6)(4, 8), (1, 5)(3, 7)\}$$

$$C4 = \{(1, 5)(2, 6), (1, 5)(4, 8), (2, 6)(3, 7), (3, 7)(4, 8)\}$$

$$C5 = \{(1, 3)(2, 8)(4, 6)(5, 7), (1, 7)(2, 4)(3, 5)(6, 8), (1, 7)(2, 8)(3, 5)(4, 6), (1, 3)(2, 4)(5, 7)(6, 8)\}$$

$$C6 = \{(1, 3, 5, 7)(2, 4, 6, 8), (1, 7, 5, 3)(2, 8, 6, 4)\}$$

$$C7 = \{(1, 3, 5, 7)(2, 8, 6, 4), (1, 7, 5, 3)(2, 4, 6, 8)\}$$

$$C8 = \{(1, 2, 7, 8, 5, 6, 3, 4), (1, 2, 3, 4, 5, 6, 7, 8), (1, 6, 3, 8, 5, 2, 7, 4), (1, 6, 7, 4, 5, 2, 3, 8)\}$$

$$C9 = \{(1, 8, 7, 6, 5, 4, 3, 2), (1, 4, 3, 6, 5, 8, 7, 2), (1, 8, 3, 2, 5, 4, 7, 6), (1, 4, 7, 2, 5, 8, 3, 6)\}$$

$$C10 = \{(1, 6, 3, 4, 5, 2, 7, 8), (1, 2, 3, 8, 5, 6, 7, 4), (1, 2, 7, 4, 5, 6, 3, 8), (1, 6, 7, 8, 5, 2, 3, 4)\}$$

$$C11 = \{(1, 4, 3, 2, 5, 8, 7, 6), (1, 4, 7, 6, 5, 8, 3, 2), (1, 8, 7, 2, 5, 4, 3, 6), (1, 8, 3, 6, 5, 4, 7, 2)\}$$

Similarly, the subgroup H of G generated by $\langle (1, 2, 3, 4, 5, 6, 7, 8), (1, 3, 5, 7)(2, 4, 6, 8), (1, 5)(2, 6)(3, 7)(4, 8) \rangle$, has the following conjugacy classes.

Conjugacy Classes of H:

$$D1 = \{\text{Id}(G)\}$$

$$D2 = \{(1, 5)(2, 6)(3, 7)(4, 8)\}$$

$$D3 = \{(1, 3, 5, 7)(2, 4, 6, 8)\}$$

$$D4 = \{(1, 7, 5, 3)(2, 8, 6, 4)\}$$

$$D5 = \{(1, 2, 3, 4, 5, 6, 7, 8)\}$$

$$D6 = \{(1, 4, 7, 2, 5, 8, 3, 6)\}$$

$$D7 = \{(1, 6, 3, 8, 5, 2, 7, 4)\}$$

$$D8 = \{(1, 8, 7, 6, 5, 4, 3, 2)\}$$

We also consider the irreducible characters ϕ (of H) listed below

Table 2.1: Irreducible Characters ϕ

Class	Size	Representative	ϕ
D1	1	Id(G)	1
D2	1	(1, 5)(2, 6)(3, 7)(4, 8)	-1
D3	1	(1, 3, 5, 7)(2, 4, 6, 8)	$-\mathbb{Z}_8^2$
D4	1	(1, 7, 5, 3)(2, 8, 6, 4)	\mathbb{Z}_8^2
D5	1	(1, 2, 3, 4, 5, 6, 7, 8)	$-\mathbb{Z}_8^3$
D6	1	(1, 4, 7, 2, 5, 8, 3, 6)	$-\mathbb{Z}_8$
D7	1	(1, 6, 3, 8, 5, 2, 7, 4)	\mathbb{Z}_8^3
D8	1	(1, 8, 7, 6, 5, 4, 3, 2)	\mathbb{Z}_8

Similarly, we consider the induced character table ϕ^G .

Table 2.2: Induced Characters ϕ^G

Class	Size	Representative	ϕ^G
C1	1	Id(G)	1
C2	1	(1, 5)(2, 6)(3, 7)(4, 8)	4
C3	2	(1, 5)(3, 7)	-4
C4	4	(1, 5)(2, 6)	0
C5	4	(1, 3)(2, 8)(4, 6)(5, 7)	0
C6	2	(1, 3, 5, 7)(2, 4, 6, 8)	0
C7	2	(1, 3, 5, 7)(2, 8, 6, 4)	0
C8	4	(1, 2, 3, 4, 5, 6, 7, 8)	0
C9	4	(1, 4, 7, 2, 5, 8, 3, 6)	0
C10	4	(1, 6, 3, 4, 5, 2, 7, 8)	0
C11	4	(1, 4, 7, 6, 5, 8, 3, 2)	0

We induce the character ϕ of H up to G to obtain the character ϕ^G of G ($\phi \uparrow_H^G$) as follows.

$$\phi_\alpha^G = \frac{n}{h_\alpha} \sum_{w \in H \cap C_\alpha} \phi(w), \text{ where } n = \frac{|G|}{|H|} = \frac{32}{8} = 4.$$

$$\phi_1^G = \frac{4}{1} \sum_{w \in H \cap C_1} \phi(1) = 4(1) = 4$$

$$\phi_2^G = \frac{4}{1} \sum_{w \in H \cap C_2} \phi(-1) = 4(-1) = -4$$

$$\phi_3^G = \frac{4}{2} \sum_{w \in H \cap C_3} \phi(0) = 2(0) = 0$$

$$\phi_4^G = \frac{4}{4} \sum_{w \in H \cap C_4} \phi(0) = 1(0) = 0$$

$$\phi_5^G = \frac{4}{4} \sum_{w \in H \cap C_5} \phi(0) = 1(0) = 0$$

$$\phi_6^G = \frac{4}{2} \sum_{w \in H \cap C_6} \phi(0) = 2(0) = 0$$

$$\phi_7^G = \frac{4}{2} \sum_{w \in H \cap C_7} \phi(0) = 2(0) = 0$$

$$\phi_8^G = \frac{4}{4} \sum_{w \in H \cap C_8} \phi(-\mathbb{Z}_8^3 + \mathbb{Z}_8^3) = 1(0) = 0$$

$$\phi_9^G = \frac{4}{4} \sum_{w \in H \cap C_9} \phi(\mathbb{Z}_8 - \mathbb{Z}_8) = 1(0) = 0$$

$$\phi_{10}^G = \frac{4}{4} \sum_{w \in H \cap C_{10}} \phi(0) = 1(0) = 0$$

$$\phi_{11}^G = \frac{4}{4} \sum_{w \in H \cap C_{11}} \phi(0) = 1(0) = 0$$

$$\text{Thus, } \phi \uparrow_H^G = \begin{pmatrix} 4 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We find the right coset representation of G over H (transversal of H in G), such that, $G = He \cup H(2, 6)(3, 7) \cup H(1, 2, 7, 4, 5, 6, 3, 8) \cup H(1, 6, 7, 4, 5, 2, 3, 8)$ and let $t_1 = e$, $t_2 = (2, 6)(3, 7)$, $t_3 = (1, 2, 7, 4, 5, 6, 3, 8)$, and $t_4 = (1, 6, 7, 4, 5, 2, 3, 8)$.

The monomial representation has generators

$$A(xx) = \begin{bmatrix} \phi(t_1xt_1^{-1}) & \phi(t_1xt_2^{-1}) & \phi(t_1xt_3^{-1}) & \phi(t_1xt_4^{-1}) \\ \phi(t_2xt_1^{-1}) & \phi(t_2xt_2^{-1}) & \phi(t_2xt_3^{-1}) & \phi(t_2xt_4^{-1}) \\ \phi(t_3xt_1^{-1}) & \phi(t_3xt_2^{-1}) & \phi(t_3xt_3^{-1}) & \phi(t_3xt_4^{-1}) \\ \phi(t_4xt_1^{-1}) & \phi(t_4xt_2^{-1}) & \phi(t_4xt_3^{-1}) & \phi(t_4xt_4^{-1}) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A(yy) = \begin{bmatrix} \phi(t_1yt_1^{-1}) & \phi(t_1yt_2^{-1}) & \phi(t_1yt_3^{-1}) & \phi(t_1yt_4^{-1}) \\ \phi(t_2yt_1^{-1}) & \phi(t_2yt_2^{-1}) & \phi(t_2yt_3^{-1}) & \phi(t_2yt_4^{-1}) \\ \phi(t_3yt_1^{-1}) & \phi(t_3yt_2^{-1}) & \phi(t_3yt_3^{-1}) & \phi(t_3yt_4^{-1}) \\ \phi(t_4yt_1^{-1}) & \phi(t_4yt_2^{-1}) & \phi(t_4yt_3^{-1}) & \phi(t_4yt_4^{-1}) \end{bmatrix}$$

$$= \begin{bmatrix} -\mathbb{Z}_{16}^6 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \mathbb{Z}_{16}^4 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{Z}_{16}^6 \end{bmatrix}$$

Since G has 8th root of unity we find that \mathbb{Z}_{17} is the smallest field which has 8th root of unity. Thus, the primitive root of \mathbb{Z}_{17} is 3, and the entries of the matrix $A(xx)$ and $A(yy)$ are powers of 3, namely, $-3^6 \equiv_{17} 2$, $3^4 \equiv_{17} 13$, and $3^6 \equiv_{17} 15$.

The permutation representation of $A(xx)$ and $A(yy)$ of the monomial representation is

$$A(xx) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

where $a_{12} = 1$, $a_{21} = 1$, $a_{34} = 1$, and $a_{43} = 1$. Therefore,

$$t_1 \rightarrow t_2,$$

$$t_2 \rightarrow t_1,$$

$$t_3 \rightarrow t_4,$$

$$t_4 \rightarrow t_3.$$

Since, the we have 4 t'_i s of order 17 we label them accordingly:

Table 2.3: Labeling for A(xx)

1	t_1	\longrightarrow	t_2	2
2	t_2	\longrightarrow	t_1	1
3	t_3	\longrightarrow	t_4	4
4	t_4	\longrightarrow	t_3	3
5	t_1^2	\longrightarrow	t_2^2	6
6	t_2^2	\longrightarrow	t_1^2	5
7	t_3^2	\longrightarrow	t_4^2	8
8	t_4^2	\longrightarrow	t_3^2	7
9	t_1^3	\longrightarrow	t_2^3	10
10	t_2^3	\longrightarrow	t_1^3	9
11	t_3^3	\longrightarrow	t_4^3	12
12	t_4^3	\longrightarrow	t_3^3	11

13	t_1^4	\longrightarrow	t_2^4	14
14	t_2^4	\longrightarrow	t_1^4	13
15	t_3^4	\longrightarrow	t_4^4	16
16	t_4^4	\longrightarrow	t_3^4	15
17	t_1^5	\longrightarrow	t_2^5	18
18	t_2^5	\longrightarrow	t_1^5	17
19	t_3^5	\longrightarrow	t_4^5	20
20	t_4^5	\longrightarrow	t_3^5	19
21	t_1^6	\longrightarrow	t_2^6	22
22	t_2^6	\longrightarrow	t_1^6	21
23	t_3^6	\longrightarrow	t_4^6	24
24	t_4^6	\longrightarrow	t_3^6	23
25	t_1^7	\longrightarrow	t_2^7	26
26	t_2^7	\longrightarrow	t_1^7	25
27	t_3^7	\longrightarrow	t_4^7	28
28	t_4^7	\longrightarrow	t_3^7	27
29	t_1^8	\longrightarrow	t_2^8	30
30	t_2^8	\longrightarrow	t_1^8	29
31	t_3^8	\longrightarrow	t_4^8	32
32	t_4^8	\longrightarrow	t_3^8	31
33	t_1^9	\longrightarrow	t_2^9	34
34	t_2^9	\longrightarrow	t_1^9	33
35	t_3^9	\longrightarrow	t_4^9	36
36	t_4^9	\longrightarrow	t_3^9	35
37	t_1^{10}	\longrightarrow	t_2^{10}	38
38	t_2^{10}	\longrightarrow	t_1^{10}	37
39	t_3^{10}	\longrightarrow	t_4^{10}	40
40	t_4^{10}	\longrightarrow	t_3^{10}	39
41	t_1^{11}	\longrightarrow	t_2^{11}	42
42	t_2^{11}	\longrightarrow	t_1^{11}	41
43	t_3^{11}	\longrightarrow	t_4^{11}	44
44	t_4^{11}	\longrightarrow	t_3^{11}	43
45	t_1^{12}	\longrightarrow	t_2^{12}	46
46	t_2^{12}	\longrightarrow	t_1^{12}	45
47	t_3^{12}	\longrightarrow	t_4^{12}	48
48	t_4^{12}	\longrightarrow	t_3^{12}	47

49	t_1^{13}	\longrightarrow	t_2^{13}	50
50	t_2^{13}	\longrightarrow	t_1^{13}	49
51	t_3^{13}	\longrightarrow	t_4^{13}	52
52	t_4^{13}	\longrightarrow	t_3^{13}	51
53	t_1^{14}	\longrightarrow	t_2^{14}	54
54	t_2^{14}	\longrightarrow	t_1^{14}	53
55	t_3^{14}	\longrightarrow	t_4^{14}	56
56	t_4^{14}	\longrightarrow	t_3^{14}	55
57	t_1^{15}	\longrightarrow	t_2^{15}	58
58	t_2^{15}	\longrightarrow	t_1^{15}	57
59	t_3^{15}	\longrightarrow	t_4^{15}	60
60	t_4^{15}	\longrightarrow	t_3^{15}	59
61	t_1^{16}	\longrightarrow	t_2^{16}	62
62	t_2^{16}	\longrightarrow	t_1^{16}	61
63	t_3^{16}	\longrightarrow	t_4^{16}	64
64	t_4^{16}	\longrightarrow	t_3^{16}	63

Therefore, $A(xx) = (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24)(25, 26)(27, 28) (29, 30)(31, 32)(33, 34)(35, 36)(37, 38)(39, 40)(41, 42)(43, 44)(45, 46)(47, 48)(49, 50)(51, 52)(53, 54) (55, 56)(57, 58)(59, 60)(61, 62)(63, 64)$.

Similarly,

$$A(yy) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 13 & 0 & 0 \\ 0 & 0 & 0 & 15 \end{bmatrix}, \text{ where } a_{11} = 2, a_{23} = 1, a_{32} = 13,$$

and $a_{44} = 15$. Therefore, $t_1 \rightarrow t_1^2$, $t_2 \rightarrow t_3$, $t_3 \rightarrow t_2^{13}$, $t_4 \rightarrow t_4^{15}$, We label the t'_i s as previously.

Table 2.4: Labeling for A(yy)

1	t_1	\longrightarrow	t_2	5
2	t_2	\longrightarrow	t_3	3
3	t_3	\longrightarrow	t_2^{13}	50
4	t_4	\longrightarrow	t_4^{15}	60
5	t_1^2	\longrightarrow	t_1^4	13
6	t_2^2	\longrightarrow	t_3^2	7
7	t_3^2	\longrightarrow	t_2^9	34
8	t_4^2	\longrightarrow	t_4^{13}	52
9	t_1^3	\longrightarrow	t_1^6	21
10	t_2^3	\longrightarrow	t_3^3	11
11	t_3^3	\longrightarrow	t_2^5	18
12	t_4^3	\longrightarrow	t_4^{11}	44
13	t_1^4	\longrightarrow	t_1^8	29
14	t_2^4	\longrightarrow	t_3^4	15
15	t_3^4	\longrightarrow	t_2	2
16	t_4^4	\longrightarrow	t_4^9	36
17	t_1^5	\longrightarrow	t_1^{10}	37
18	t_2^5	\longrightarrow	t_3^5	19
19	t_3^5	\longrightarrow	t_2^{14}	54
20	t_4^5	\longrightarrow	t_4^7	28
21	t_1^6	\longrightarrow	t_1^{12}	45
22	t_2^6	\longrightarrow	t_3^6	23
23	t_3^6	\longrightarrow	t_2^{10}	38
24	t_4^6	\longrightarrow	t_4^5	20
25	t_1^7	\longrightarrow	t_1^{14}	53
26	t_2^7	\longrightarrow	t_3^7	27
27	t_3^7	\longrightarrow	t_2^6	22
28	t_4^7	\longrightarrow	t_4^3	12
29	t_1^8	\longrightarrow	t_1^{16}	61
30	t_2^8	\longrightarrow	t_3^8	31
31	t_3^8	\longrightarrow	t_2^2	6
32	t_4^8	\longrightarrow	t_4	4
33	t_1^9	\longrightarrow	t_1	1
34	t_2^9	\longrightarrow	t_3^9	35
35	t_3^9	\longrightarrow	t_2^{15}	58
36	t_4^9	\longrightarrow	t_4^{16}	64

37	t_1^{10}	\longrightarrow	t_1^3	9
38	t_2^{10}	\longrightarrow	t_3^{10}	39
39	t_3^{10}	\longrightarrow	t_2^{11}	42
40	t_4^{10}	\longrightarrow	t_4^{14}	56
41	t_1^{11}	\longrightarrow	t_1^5	17
42	t_2^{11}	\longrightarrow	t_3^{11}	43
43	t_3^{11}	\longrightarrow	t_2^7	26
44	t_4^{11}	\longrightarrow	t_4^{12}	48
45	t_1^{12}	\longrightarrow	t_1^7	25
46	t_2^{12}	\longrightarrow	t_3^{12}	47
47	t_3^{12}	\longrightarrow	t_2^3	10
48	t_4^{12}	\longrightarrow	t_4^{10}	40
49	t_1^{13}	\longrightarrow	t_1^9	33
50	t_2^{13}	\longrightarrow	t_3^{13}	51
51	t_3^{13}	\longrightarrow	t_2^{16}	62
52	t_4^{13}	\longrightarrow	t_4^8	32
53	t_1^{14}	\longrightarrow	t_1^{11}	41
54	t_2^{14}	\longrightarrow	t_3^{14}	55
55	t_3^{14}	\longrightarrow	t_2^{12}	46
56	t_4^{14}	\longrightarrow	t_4^6	24
57	t_1^{15}	\longrightarrow	t_1^{13}	49
58	t_2^{15}	\longrightarrow	t_3^{15}	59
59	t_3^{15}	\longrightarrow	t_2^8	30
60	t_4^{15}	\longrightarrow	t_4^4	16
61	t_1^{16}	\longrightarrow	t_1^{15}	57
62	t_2^{16}	\longrightarrow	t_3^{16}	63
63	t_3^{16}	\longrightarrow	t_2^4	14
64	t_4^{16}	\longrightarrow	t_4^2	8

Therefore, $A(yy) = (1, 5, 13, 29, 61, 57, 49, 33)(2, 3, 50, 51, 62, 63, 14, 15)(4, 60, 16, 36, 64, 8, 52, 32)(6, 7, 34, 35, 58, 59, 30, 31)(9, 21, 45, 25, 53, 41, 17, 37)(10, 11, 18, 19, 54, 55, 46, 47)(12, 44, 48, 40, 56, 24, 20, 28)(22, 23, 38, 39, 42, 43, 26, 27)$. In MAGMA we confirm if $A(xx)$ and $A(yy)$ generate a group isomorphic to the group G .

```
S:=Sym(64);
a:=S!(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)
(17,18)(19,20)(21,22)(23,24)(25,26)(27,28)(29,30)(31,32)
(33,34)(35,36)(37,38)(39,40)(41,42)(43,44)(45,46)(47,48)
(49,50)(51,52)(53,54)(55,56)(57,58)(59,60)(61,62)(63,64);

b:=S!(1,5,13,29,61,57,49,33)(2,3,50,51,62,63,14,15)
(4,60,16,36,64,8,52,32)(6,7,34,35,58,59,30,31)
(9,21,45,25,53,41,17,37)(10,11,18,19,54,55,46,47)
(12,44,48,40,56,24,20,28)(22,23,38,39,42,43,26,27);

HH:=sub<S|a,b>;
IsIsomorphic(HH,G);

true Mapping from: GrpPerm: HH to GrpPerm: G
Composition of Mapping from: GrpPerm: HH to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: G

FPGGroup(G);
Finitely presented group on 2 generators
Relations
$.1^2 = Id($)
($.2 * $.1 * $.2)^2 = Id($)
$.2^8 = Id($)
($.2 * $.1 * $.2^-1 * $.1)^2 = Id($)

K<x,y>:=Group<x,y|x^2,(y * x * y)^2,y^8,(y * x * y^-1 * x)^2>;
#K;
32
```


Thus, the work is correct.

Let $G = \langle x, y \rangle$ such that $x = A(xx)$ and $y = A(yy)$ (as above) then the presentation for the monomial matrix is given by $\langle x, y \mid x^2, (xyx)^2, y^8, (yxy^{-1}x)^2 \rangle$ and has order 32. We now wish to build a monomial progenitor from this presentation so we proceed as shown in the following steps.

Step 1:

t commutes with the normalizer of $\langle t \rangle$ in G . We find $\langle t_1 \rangle = \{t_1, t_1^2, t_1^3, t_1^4, t_1^5, \dots, t_1^{16}, e\}$. The normalizer of this group is $\{n \in G \mid \{t_1, t_1^2, \dots, t_1^{16}, e\}^n = \{t_1, t_1^2, \dots, t_1^{16}\}\}$, by the labeling of the automorphisms (see tables above) we have $t_1 = 1, t_1^2 = 5, \dots, t_1^{16} = 61$.

We compute the generators of the normalizer of $\langle t_1 \rangle$ in G .

```
>Stabilizer(G,{1,5,9,13,17,21,25,29,33,37,41,45,49,53,57,61});
Permutation group acting on a set of cardinality 64
Order = 8 = 2^3
(1, 29, 49, 5, 61, 33, 13, 57)(2, 51, 14, 3, 62, 15, 50, 63)
(4, 36, 52, 60, 64, 32, 16, 8)(6, 35, 30, 7, 58, 31, 34, 59)
(9, 25, 17, 21, 53, 37, 45, 41)(10, 19, 46, 11, 54, 47, 18, 55)
(12, 40, 20, 44, 56, 28, 48, 24)(22, 39, 26, 23, 42, 27, 38, 43).
>yy^3;
(1, 29, 49, 5, 61, 33, 13, 57)(2, 51, 14, 3, 62, 15, 50, 63)
(4, 36, 52, 60, 64, 32, 16, 8)(6, 35, 30, 7, 58, 31, 34, 59)
(9, 25, 17, 21, 53, 37, 45, 41)(10, 19, 46, 11, 54, 47, 18, 55)
(12, 40, 20, 44, 56, 28, 48, 24)(22, 39, 26, 23, 42, 27, 38, 43)
```

We find the single permutation $(1, 29, 49, 5, 61, 33, 13, 57)(2, 51, 14, 3, 62, 15, 50, 63)(4, 36, 52, 60, 64, 32, 16, 8)(6, 35, 30, 7, 58, 31, 34, 59)(9, 25, 17, 21, 53, 37, 45, 41)(10, 19, 46, 11, 54, 47, 18, 55)(12, 40, 20, 44, 56, 28, 48, 24)(22, 39, 26,$

23, 42, 27, 38, 43), generates the normalizer of G in $\langle t_1 \rangle$ and notice that the permutation is represented as yy^3 . This permutation takes 1 to 29, then by the labeling of the automorphisms we have $t_1^{y^3} = t_1^8$ (note $yy \sim y$).

Step 2:

We find the point stabilizer of 1 in G , along with its orbits.

```
>Stabilizer(G,1);
Permutation group acting on a set of cardinality 8
  Order = 4 = 2^2
    (2, 6)(3, 7)
    (3, 7)(4, 8)
> Orbits(Stabilizer(G,1));
[
  GSet{@ 1 @},
  GSet{@ 5 @},
  GSet{@ 2, 6 @},
  GSet{@ 3, 7 @},
  GSet{@ 4, 8 @}
]
```

Thus, there are 2 generators and 5 orbits. The permutations in G which take 1 to 1, 1 to 5, 1 to 2, 1 to 3, and 1 to 4 are given in words in the following code.

```
> 1^(xx*yy);
3
> 1^(yy);
5
> 1^(xx);
2
> 1^(xx*yy*xx);
4
```

Thus, the permutation represented by $xx*yy$ is a permutation which takes t_1 to t_3 .

Step 3:

Now, we are ready to build the progenitor for G and test it by the Grindstaff

Lemma (**section 1.1**).

```
> FPGroup(G);
Finitely presented group on 2 generators
Relations
  $.1^2 = Id($)
  ($.2 * $.1 * $.2)^2 = Id($)
  $.2^8 = Id($)
  ($.2 * $.1 * $.2^-1 * $.1)^2 = Id($)
>
> G<x,y,t>:=Group<x,y,t| x^2,(y * x * y)^2,y^8,(y * x * y^-1 * x)^2,t^17,
      t^(y^3)=t^8,(t,t^(x*y)),(t,t^y),(t,t^x),(t,t^(x*y*x))>;
> #G;
2672672
>
> (17)^4*32;
2672672
```

We have found a symmetric presentation for the progenitor $17^{*4} :_m (2^3 : \bullet 4)$ to be $\langle x, y, t \mid x^2, (xyx)^2, y^8, (xyx^{-1}x)^2, t^{17}, t^{y^3} = t^8 \rangle$. We obtain our first order relations as previously done in **section 1.2**. Thus, the first order relations are $(y^4t)^a, ((xyxy^{-1})t)^b, (xt)^c, ((xy^2)t)^d, (yt)^e, (y^3t)^f, (y^2t)^g, ((xy)^2t)^h, ((xy)t)^i, ((xyx^{-1})t)^j$. We factor the progenitor by these relations to obtain finite images (see **Chapter 3**). Thus, the final symmetric presentation is given by $G\langle x,y,t \mid x^2,(xyx)^2, y^8, (xyx^{-1}x)^2,t^{17}, t^{y^3} = t^8,(y^4t)^a, (xyxy^{-1}t)^b, (xt)^c, (xy^2t)^d, (yt)^e, (y^3t)^f, (y^2t)^g, ((xy)^2t)^h, (xyt)^i, (xyx^{-1}t)^j \rangle$.

Chapter 3

Finite Images of Progenitors

3.1 Progenitor $2^{*6} : (3^2 : 2^2)$

$$N = \langle x, y, z \rangle \leq S_6$$

$$x \sim (2, 4, 6); y \sim (1, 5)(2, 4); z \sim (1, 4)(2, 5)(3, 6); \text{ and}$$

$$t \sim t_1$$

$$G \langle x, y, z, t \rangle := \text{Group} \langle x, y, z, t \mid x^3, y^2, z^2, (x^{-1}y)^2, (yz)^2, x^{-1}zx^{-1}zxzxz, t^2, (t, xyzxz), (t, x), \\ (zt)^a, (yzt)^b, (yt)^c, (yt^{(zz)})^d, ((xz)^2t)^e, (xxz^{-1}zt)^f, (zxzt)^g, (zxzt^{(yz)})^h, (zxt)^i, (yzt)^j \rangle;$$

Table 3.1: Finite Images of $2^{*6} : (3^2 : 2^2)$

f	g	h	i	j	Order of G	Shape of G
0	3	0	6	6	576	$2^4 : (3^2 : 2^2)$
0	3	6	10	10	10,080	$S_7 \times 2$
0	0	0	0	4	432	$3^3 : 2^4$
4	0	0	8	10	760,320	$\text{Aut}(M_{12}) : 2^2$
,4	0	0	0	8	7,741,440	$(8 \times 2^2) : (\text{PGL}(3, 4) : 2^2)$

3.2 Progenitor $2^{*6} : A_5$

$$N = \langle x, y \rangle \leq S_6$$

$$x \sim (1, 2, 3, 4, 6); y \sim (1, 4)(5, 6); t \sim t_1$$

$$G = \langle x, y, t \rangle := \text{Group} \langle x, y, t \mid x^5, y^2, (yx^{-1})^3, t^2, (t, (yx^2)^2), (t, xyx^{-1}), (yt)^a, (yt^x)^b, (yt^{(x^{-1}y)})^c, (yxt)^d, (yxt^x)^e, (xt)^f, (xt^{(x^{-1}y)})^g, (x^2t^{(x^{-1}y)})^h, (x^2t)^i \rangle;$$

Table 3.2: Finite Images of $2^{*6} : A_5$

e	f	g	h	i	Order of G	Shape of G
0	0	0	0	7	34,440	PSL(2,41)
9	9	0	0	0	3,420	PSL(2,19)
0	6	0	0	10	241,920	PGL(3,4):6
8	8	0	10	0	190,080	$M_{12} \times 2$
0	0	10	10	4	1,320	PSL(2,11)

3.3 Progenitor $2^{*12} : PGL(2, 9)$

$$N = \langle x, y \rangle \leq S_{12};$$

$$x \sim (2, 4, 10)(3, 5, 7)(6, 12, 8); y \sim (1, 10, 9, 2, 7, 12, 11, 8, 5, 4)(3, 6); t \sim t_1$$

$$G = \langle x, y, t \rangle := \text{Group} \langle x, y, t \mid x^3, (xy^{-1})^2, y^{10}, (x^{-1}y^{-2})^4, y^3x^{-1}y^{-4}xy^3xy^{-4}x^{-1}y^2, t^2, (t, x), (t, y^2xy^{-2}), (y^5t^{(y^{-2}x^{-1})})^a, (y^5t)^b, ((yxy)^2t^{y^{-2}x^{-1}})^c, ((yxy)^2t)^d, (xt)^e, (xt^{(yx)})^f, (xt^{y^{-2}x^{-1}})^g, (xt^{y^{-2}x^{-1}y})^h, (yxyt^{y^{-2}x^{-1}})^i, (yxyt)^j, (y^2t^{y^{-2}x^{-1}})^k, (y^2t)^l, (y^{-4}t^{y^{-2}x^{-1}})^m, (y^{-4}t)^n, (yxy^2t^{y^{-1}})^o, (yxy^2t)^p, (xyxy^3xy^{-3}t^{y^{-1}})^q, (xyxy^3xy^{-3}t)^r, (yt^{y^{-2}x^{-1}})^s, (yt)^u, (y^3t^{y^{-2}x^{-1}})^v, (y^3t)^w \rangle;$$

Table 3.3: Finite Images of $2^{*12} : PGL(2, 9)$

r	s	u	v	w	Order of G	Shape of G
0	10	0	5	8	252,000	U(3,5):2

3.4 Progenitor $2^{*100} : (A_5 \times D_{10})$

$$N = \langle x, y \rangle \leq S_{100};$$

$$\begin{aligned} x \sim & (1, 96, 19, 5, 94)(2, 71, 22, 13, 36)(3, 33, 86, 9, 34)(4, 37, 49, 56, 74)(6, 72, \\ & 51, 38, 75)(7, 54, 43, 98, 20)(8, 83, 40, 31, 45)(10, 70, 52, 61, 30)(11, 15, 91, 23, \\ & 41)(12, 35, 59, 42, 57)(14, 60, 89, 87, 66)(16, 90, 100, 62, 46)(17, 99, 53, 58, 65)(18, \\ & 32, 78, 67, 95)(21, 84, 64, 81, 97)(24, 55, 44, 93, 92)(25, 48, 50, 80, 76)(26, 73, 88, \\ & 63, 29)(27, 68, 82, 28, 85)(39, 69, 79, 47, 77); \end{aligned}$$

$$\begin{aligned} y \sim & (1, 61, 42, 21, 100, 9)(2, 18, 50, 84, 59, 53)(3, 88, 76, 67, 96, 86)(4, 8, 55, 15, \\ & 60, 29)(5, 98, 38, 22, 16, 79)(6, 27)(7, 32, 36, 49, 17, 93)(10, 31, 83, 74, 72, 85)(11, \\ & 68, 75, 43, 39, 89)(12, 92, 65)(13, 51, 56)(14, 62, 97, 82, 41, 26)(19, 78, 20)(23, \\ & 25, 73)(24, 57, 52, 64, 48, 91)(28, 81, 70)(30, 94, 69, 54, 44, 45)(34, 90, 71, 99, 37, \\ & 63)(35, 58)(46, 66, 47)(77, 87)(80, 95); \end{aligned}$$

$$t \sim t_1$$

$$\begin{aligned} G \langle x, y, t \rangle := & \text{Group} \langle x, y, t | x^5, y^6, (x^{-1} * y^{-1})^2, x^{-1} * y^{-3} * x * y^{-1} * x * y^3 * x^{-1} * \\ & y, t^2, (t, x * y^2 * x^{-2} * y), ((x^{-1} * y * x * y^{-2} * x^2) * t^{(x * y^{-1} * x^2 * y^{-1})})^a, ((x^{-1} * y * x * y^{-1} * \\ & x * y * x^{-1} * y) * t)^b, ((x^2 * y * x^{-1}) * t^{(x^{-2})})^c, ((x * y^{-1} * x * y) * t^{(x * y^2)})^d, (((x * y^{-2})^2) * \\ & t)^e, ((y * x^2 * y^3 * x^{-2} * y^2) * t)^f, (((x * y^{-1})^2) * t)^g, (((x * y^{-1})^4) * t)^h, ((x) * t)^i \rangle; \end{aligned}$$

Table 3.4: Finite Images of $2^{*100} : (A_5 \times D_{10})$

f	g	h	i	Order of G	Shape of G
2	10	0	3	1,320	$\text{PSL}(2,11) \times 2$
0	10	6	3	79,200	$\text{PSL}(2,11) \times A_5 \times 2$

3.5 Progenitor $2^{*12} : S_4$

$$N = \langle x, y \rangle \leq S_{12};$$

$$x \sim (1, 2)(3, 5)(4, 6)(7, 9)(8, 10);$$

$$y \sim (1, 3, 6, 12)(2, 4, 7, 10)(5, 8, 11, 9);$$

$$t \sim t_1$$

$$G = \langle x, y, t \rangle := \text{Group} \langle x, y, t | x^2, y^4, (x * y^{-1})^3, t^2, (t, x^y), ((y^2) * t^x)^a, ((y^2) * t)^b, ((y * x * y^{-1}) * t)^c, ((y * x * y^{-1}) * t^{(y^2)})^d, ((y * x * y^{-1}) * t^x)^e, ((y * x * y^{-1}) * t^{(y * x)})^f, ((y * x) * t)^g, ((y * x) * t^x)^h, ((y * x) * t^y)^i, ((y * x) * t^{(x * y^2)})^j, (y * t)^k, (y * t^x)^l, (y * t^{(y * x)})^m \rangle;$$

Table 3.5: Finite Images of $2^{*12} : S_4$

i	j	k	l	m	Order of G	Shape of G
0	7	0	2	9	504	PSL(2,8)
0	9	0	2	9	3,420	PSL(2,19)
3	7	0	5	5	20,160	PSL(3,4)
0	4	0	3	8	23,040	$2^6 : A_6$
0	4	0	5	5	7,200	$S_5 \times A_5$
0	6	0	3	6	46,800	$\text{PSL}(2,25) \times S_3$
0	8	0	2	10	4,320	$S_6 \times 6$
2	0	0	5	0	3,960	$\text{PGL}(2,11) \times 3$
3	5	0	0	7	2,520	A_7
2	0	0	6	4	240	$S_5 \times 2$
3	5	0	6	8	1920	$S_5 : 2^4$
2	0	0	7	7	2,184	$\text{PGL}(2,13)$
2	6	0	0	8	672	$\text{PGL}(2,7) \times 2$
2	7	0	0	0	24,360	$\text{PGL}(2,29)$
3	0	0	5	5	40,320	$L(3,4) = M_{12} \times 2$
3	0	0	7	5	178,920	PSL(2,71)
3	4	0	0	9	2,448	PSL(2,17)
4	0	0	8	4	1,062,720	$\text{PGL}(2,81) \times 2$
5	4	0	3	4	360	A_6
7	7	0	0	4	2,384,928	$\text{PGL}(2,13) \times \text{PSL}(2,13)$

3.6 Monomial Progenitor $17^{*4} :_m (2^3 :^\bullet 4)$

$$N = \langle x, y \rangle \leq S_{64};$$

$$\begin{aligned} x \sim A(xx) = & (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16) (17,18)(19,20)(21,22) \\ & (23,24)(25,26)(27,28)(29,30)(31,32) (33,34)(35,36)(37,38)(39,40)(41,42)(43,44) \\ & (45,46)(47,48) (49,50)(51,52)(53,54)(55,56)(57,58)(59,60)(61,62)(63,64) \end{aligned}$$

$$\begin{aligned} y \sim A(xx) = & (1,5,13,29,61,57,49,33)(2,3,50,51,62,63,14,15) (4,60,16,36,64,8,52,32) \\ & (6,7,34,35,58,59,30,31) (9,21,45,25,53,41,17,37)(10,11,18,19,54,55,46,47) \\ & (12,44,48,40,56,24,20,28)(22,23,38,39,42,43,26,27) \end{aligned}$$

$$t \sim t_1$$

$$\begin{aligned} G \langle x, y, t \mid & x^2, (xyx)^2, y^8, (xyx^{-1}x)^2, t^{17}, t^{y^3} = t^8, (y^4t)^a, (xyxy^{-1}t)^b, (xt)^c, (xy^2t)^d, \\ & (yt)^e, (y^3t)^f, (y^2t)^g, ((xy)^2t)^h, (xyt)^i, (xyx^{-1}t)^j \rangle \end{aligned}$$

Table 3.6: Finite Images of $17^{*4} :_m (2^3 :^\bullet 4)$

d	h	i	j	Order of G	Shape of G
4	10	4	5	16320	$\text{PGL}(2,16):2 = \text{Aut}(\text{PSL}(2,16))$
4	10	4	10	65,280	$\text{Aut}(\text{PSL}(2,16)) \times 2^2$

Chapter 4

Isomorphism Types

4.1 Preliminaries

Definition 4.1.1. (abelian) A pair of elements a and b in a group **commutes** if $a*b=b*a$. A group is **abelian** if every pair of its elements commutes. [Rot95]

Definition 4.1.2. (homomorphism) Let $(G,*)$ and (H,\circ) be groups. A function $f:G\leftrightarrow H$ is a **homomorphism** if, for all $a, b \in G$, $f(a*b) = f(a) \circ f(b)$. [Rot95]

Definition 4.1.3. (isomorphism) An **isomorphism** is a homomorphism that is also a bijection. We say that G is **isomorphic** to H , denoted $G\cong H$, if there exists an isomorphism $f:G \leftrightarrow H$. [Rot95]

Theorem 4.1.4. Let p be a prime. A group G of order p^n is cyclic if and only if it is an abelian group having a unique subgroup of order p . [Rot95]

Definition 4.1.5. (normal subgroup) A subgroup $K \leq G$ is a **normal subgroup**, denoted by $K \trianglelefteq G$, if $gKg^{-1} = K$ for every $g \in G$. [Rot95]

Theorem 4.1.6. (First Isomorphism Theorem) Let $f: G \rightarrow H$ be a homomorphism with kernel K . Then K is a normal subgroup of G and $G/K \cong \text{im}(f)$. [Rot95]

Theorem 4.1.7. (Second Isomorphism Theorem) Let N and T be subgroups of G with N normal. Then $N \cap T$ is normal in T and $T/(N \cap T) \cong NT/N$. [Rot95]

Theorem 4.1.8. (Third Isomorphism Theorem) Let $K \leq H \leq G$, where both K and H are normal subgroups of G . Then H/K is a normal subgroup of G/K and $(G/K)(H/K) \cong G/H$. [Rot95]

Theorem 4.1.9. (Correspondence Theorem) Let $K \trianglelefteq G$ and let $\pi: G \rightarrow G/K$ be the natural map. Then $S \mapsto \pi(S) = S/K$ is a bi-jection from the family of all those subgroups S of G which contain K to the family of all the subgroups of G/K . Moreover, if we denote S/K by S^* , then: (i) $T \leq S$ if and only if $T^* \leq S^*$, and then $[S:T] = [S^*:T^*]$; and (ii) $T \trianglelefteq S$ if and only if $T^* \trianglelefteq S^*$, and then $S/T \cong S^*/T^*$. [Rot95]

Definition 4.1.10. (maximal normal subgroup) A subgroup $H \leq G$ is a **maximal normal subgroup** of G if there is no normal subgroup N of G with $H < N < G$. [Rot95]

Definition 4.1.11. (simple) A group $G \neq 1$ is **simple** if it has no normal subgroups other than G and itself. [Rot95]

Definition 4.1.12. (direct product) If H and K are groups, then their **direct product**, denoted by $H \times K$, is the group with elements all ordered pairs (h, k) , where $h \in H$ and $k \in K$, and with the operation $(h, k)(h', k') = (hh', kk')$. [Rot95]

Theorem 4.1.13. (Jordan-Holder Theorem) Every two composition series of a group G are equivalent.

Suppose that the finite group G has two composition series

$$G = B_0 > B_1 > \dots > B_n = \{1\} \text{ and } G = C_0 > C_1 > \dots > C_m = \{1\}.$$

Then $n = m$ and the lists of composition factors for the two series are identical in the sense that if $|H| \leq |G|$ and $\Phi(H) = \{i \geq 1 : B_{i-1}/B_i \cong H\}$ and $\Psi(H) = \{i \geq 1 : C_{i-1}/C_i \cong H\}$ then $\Phi(H) = \Psi(H)$. [Rot95]

Definition 4.1.14. (semi-direct product) A group G is a **semi-direct product** of K by Q , denoted by $G = K \rtimes Q$, if $K \triangleleft G$ and K has a complement $Q_1 \cong Q$. One also says that G splits over K . [Rot95]

Definition 4.1.15. (mixed-extension) If G is an extension of an abelian group not equal to the center of G , then this is called a **mixed extension**. [Rot95]

Definition 4.1.16. (normal subgroup in composition series) A normal sub-

group N of a group G is called a maximal normal subgroup of G if

(a) $N \neq G$

(b) whenever $N \leq M \triangleleft G$ then either $M = N$ or $M = G$.

By the Correspondence Theorem, if $N \triangleleft G$ and $N \neq G$ then every normal subgroup of G/N corresponds to a normal subgroup of G containing N . So a normal subgroup N is maximal if and only if G/N is simple.

Definition 4.1.17. (Composition series) Given a group G , a **composition series** for G of length n is a sequence of subgroups $G = B_0 > B_1 > \dots > B_n = 1_G$ such that

(i) $B_i \triangleleft B_{i-1}$ for $i = 1, \dots, n$.

(ii) B_{i-1}/B_i is simple for $i = 1, \dots, n$. In particular, B_i is a maximal normal subgroup of G and B_{i-1} is simple. The (isomorphism classes of the) quotient groups B_i/B_{i-1} are called composition factors of G .

Example 4.1.18.

S_4 has the following composition series of length 4, where K is the Klein group $\{(1), (12)(34), (13)(24), (14)(23)\}$.

$$S_4 > A_4 > K > \{(12)(34)\} > \{1\}$$

We know that $A_4 \triangleleft S_4$; the composition factor $S_4/A_4 \cong C_2$.

We have seen that $K \triangleleft A_4$; and $A_4/K \cong C_3$

All subgroups of K are normal in K , because K is abelian.

Both $K/\{(12)(34)\}$ and $\{(12)(34)\}/\{1\}$ are isomorphic to C_2 .

So the composition factors of S_4 are C_2 (three times) and C_3 (once).

Example 4.1.19.

If G is simple then its only composition series is $G > \{1\}$, of length 1.

Example 4.1.20.

$(\mathbb{Z}, +)$ has no composition series. If $H \leq \mathbb{Z}$ then H is cyclic of infinite order.

If $H = \langle x \rangle$ then $\langle 2x \rangle$ is a subgroup of H with $\{0\} \neq \langle 2x \rangle \neq H$, and $\langle 2x \rangle \triangleleft H$ because H is abelian. So H is not simple. If $B_0 > B_1 > \dots > B_n$ is a composition series then B_{n-1} is simple, so there can be no composition series.

Theorem 4.1.21: Every finite group G has a composition series.

Example 4.1.22. (Composition series for Q_8 and D_8)

Q_8 :

All three possible compositions series for Q_8 are valid and are the following:

$$1 \leq \langle -1 \rangle \leq \langle i \rangle \leq Q_8$$

$$1 \leq \langle -1 \rangle \leq \langle j \rangle \leq Q_8$$

$$1 \leq \langle -1 \rangle \leq \langle k \rangle \leq Q_8$$

The respective composition factors being $\mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_2$.

D_8 :

There are 7 composition series for D_8 listed below:

$$1 \leq \langle s \rangle \leq \langle s, r^2 \rangle \leq D_8$$

$$1 \leq \langle r^2 s \rangle \leq \langle s, r^2 \rangle \leq D_8$$

$$1 \leq \langle r^2 \rangle \leq \langle s, r^2 \rangle \leq D_8$$

$$1 \leq \langle r^2 \rangle \leq \langle r \rangle \leq D_8$$

$$1 \leq \langle r^2 \rangle \leq \langle rs, r^2 \rangle \leq D_8$$

$$1 \leq \langle rs \rangle \leq \langle rs, r^2 \rangle \leq D_8$$

$$1 \leq \langle r^3 s \rangle \leq \langle rs, r^2 \rangle \leq D_8$$

Although $D_8 \not\cong Q_8$, they have the same composition factors.

4.2 Introduction

In this chapter we wish to identify groups as one of the isomorphism types. The first is a direct product (see **section 4.1**), the second is a semi-direct product (see **section 4.1**), and the third is called a mixed extension (see **section 4.1**). The process for which we can identify each of these types is very similar. We begin by outlining the process and start with applying the Jordan-Holder Theorem (**section 4.1**).

Given a composition series of G :

$$G = G_0 \geq G_1 \geq \dots \geq G_{i-1} \geq G_i = 1, \text{ for } i = 0, 1, \dots, n.$$

Note that a composition series is a normal series and the G_i 's are normal in G_{i+1} . Also, the composition series of G has composition factors of the form G_{i-1}/G_i , where the composition factors are simple since G_i is maximal normal subgroup of G_{i-1} . We go about solving the extension problem by applying the Jordan-Holder Theorem, where we wish to solve for all factors G_{i-1}/G_i of the composition series of G , until we obtain all of G . [Rot 97]

For example, let $G = G_0 \geq G_1 \geq G_2 \geq G_3 = 1$, be a composition series for G . We wish to classify the isomorphism type of G so we have:

1. G_2 is an extension of G_3 by G_2/G_3 . Thus, $G_2 = G_3 / 1 \quad G_2/G_3 = G_2/1$
thus, we know what G_2 is.
2. G_1 is an extension of G_2 by G_1/G_2 . Thus, $G_1 = G_2 \quad G_1/G_2$.
($G_2 \geq G_1$, G_1 is an extension of G_2 by G_1/G_2)

3. Now we solve the extension problem and find G_1 .

G is an extension of G_1 by G/G_1 . Thus, $G = G_1 \rtimes G/G_1$.

We solve this extension:

i. composition products: $G = G/G_1 \rtimes G_1/G_2 \rtimes G_2$ (from 1 above).

ii. looking at $G_1/G_2 \rtimes G_2$ we can find G_1 from here. Now, $G_1 = G_2 \rtimes G_1/G_2$ (by 2 above). Thus, $G = G_1 \rtimes G/G_1$.

Thus, this is how we will proceed to identify the isomorphism type of G in the following examples.

4.3 $3^2 : 2^2$

We wish to identify the Isomorphism type of N . Let N be a transitive group on 6 letters, where N is a group of order 36 generated by $x \sim (2,4,6)$, $y \sim (1, 5)(2, 4)$, and $z \sim (1, 4)(2, 5)(3, 6)$. The normal lattice of N is given below.

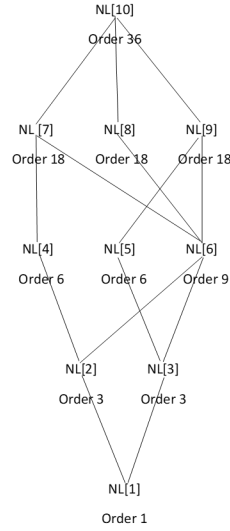


Figure 4.1: Normal Lattice of $3^2 : 2^2$

The largest normal abelian subgroup of N is $NL[6]$. $NL[6] = \langle A, B \rangle$ is of order 9, where $A \sim (2,4,6)$ and $B \sim (1,3,5)(2,6,4)$. It is possible that $NL[6]$ can be isomorphic to 3×3 or $\mathbb{Z}_9 = 9$. However, we find that $NL[6] \cong 3^2$ since it is generated by two elements of order 3. Furthermore, N is not a direct product of $NL[6]$ because N does not have a normal subgroup of order 4. Therefore, N is a semi-direct product of $NL[6]$.

We factor N by $NL[6]$ such that, N is an extension of $NL[6]$ by $N/NL[6]$. Thus, the extension is isomorphic to 2^2 and $N/NL[6] = \langle D, E \rangle = 2^2$ where $D \sim (1, 2)(3, 4)$, $E \sim (1, 3)(2, 4)$. Now, conjugating every generator of $NL[6]$ by D and E . That is, we compute A^D , A^E , B^D , and B^E . Thus, the presentation of N is $\langle a, b, d, e \mid a^3, b^3, (a,b), d^2, e^2, (de)^2, a^d = a^2, a^e = ab, b^d = b^2, b^e = b^2 \rangle$. Therefore, N is the semi-direct product $3^2:2^2$.

4.4 $4^2 : 2$

We wish to find the Isomorphism type of the transitive group N on 8 letters. N is a group of order 32 and is generated by $x \sim (1,2,3,8)$ and $y \sim (1,5)(2,6)(3,7)(4,8)$. The Normal Lattice of N is given below.

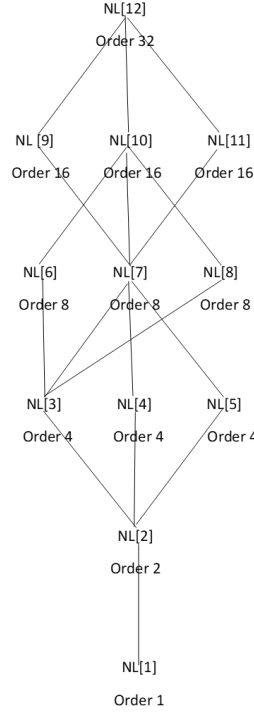


Figure 4.2: Normal Lattice of $4^2 : 2$

The largest normal abelian subgroup of N is $NL[9]$, which is of order 16. This means $NL[9]$ can be isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4 = 4 \times 4$, $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = 4 \times 2 \times 2$, or $\mathbb{Z}_{16} = 16$. Investigating further we find $NL[9] \cong \langle A, B \rangle = \langle (1,8,3,2)(4,5,6,7), (1,2,3,8) \rangle$. Thus, $NL[9]$ is generated by two elements of order 4, thus $NL[9] \cong \mathbb{Z}_4 \times \mathbb{Z}_4 = 4 \times 4 = 4^2$. The presentation for $NL[9]$ is given by $\langle a, b \mid a^4, b^4, (a,b) \rangle$.

Since $NL[9]$ is an abelian subgroup of order 16 and N is of order 32; if N had a normal subgroup of order 2, N would be a direct-product. However, N does not have a normal subgroup of order 2 which is not contained in $NL[9]$. Therefore, N is a semi-direct product of $NL[9]$. Factoring N by $NL[9]$ we see that N is an extension of $NL[9]$ by $N/NL[9]$. Furthermore, $N/NL[9] = \langle C \rangle = \mathbb{Z}_2$ where $C \sim (1,5)(2,6)(3,7)(4,8)$ of order 2. Conjugating every generator of $NL[9]$ by C . That is, we compute A^C , and B^C .

$$A^C = (1, 8, 3, 2)(4, 5, 6, 7)^{(1,5)(2,6)(3,7)(4,8)} = (5,4,7,6)(8,1,2,3) = A^3$$

$$B^C = (1, 2, 3, 8)^{(1,5)(2,6)(3,7)(4,8)} = (5,6,7,4) = AB .$$

Thus, $A^C = A^3$ and $B^C = AB$ and the complete presentation for N is

$$\langle a, b, c \mid a^4, b^4, c^2, (a,b), a^c = a^3, b^c = ab \rangle .$$

Therefore, N is the semi-direct product $4^2:2$.

4.5 $2^3 : A_4$

We wish to find the isomorphism type of a transitive group N on 8 letters, which is of order 96 and is generated by $t \sim (1, 8)(2, 3)(4, 5)(6, 7)$, $w \sim (1, 3)(2, 8)(4, 6)(5, 7)$, $x \sim (1, 5)(2, 6)(3, 7)(4, 8)$, $y \sim (1, 2, 3)(4, 6, 5)$, and $z \sim (2, 5)(3, 4)$. The Normal Lattice of N is given below:

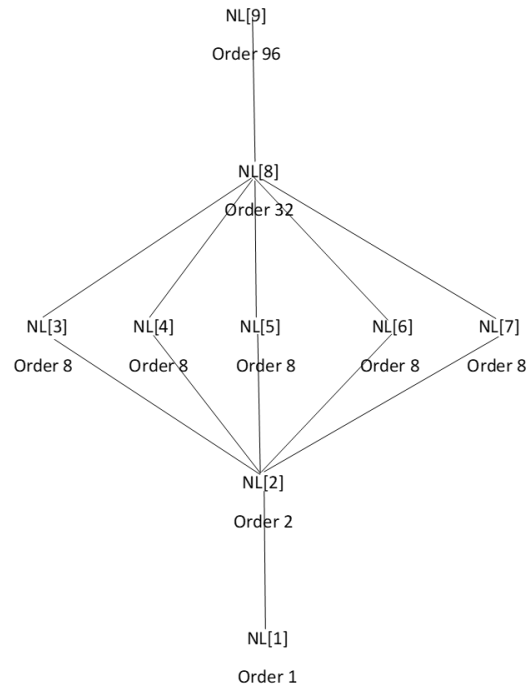


Figure 4.3: Normal Lattice of $2^3 : A_4$

The largest normal abelian subgroup of N is $NL[5]$, which has order 8. This implies $NL[5]$ can be isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, or \mathbb{Z}_8 . However, since $NL[5]$ is generated by $A \sim (2, 5)(3, 4)$, $B \sim (1, 6)(2, 5)$, and $C \sim (1, 6)(2,$

5)(3, 4)(7, 8) then $NL[5] \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = 2^3$. We identify that N is not a direct product of $NL[5]$ because the only normal subgroup in N of order 2 is contained in $NL[5]$. Therefore, N is a semi-direct product of $NL[5]$. Thus, the Presentation for $NL[5]$ is $\langle a, b, c \rangle = \langle a, b, c \mid a^2, b^2, c^2, (a,b), (a,c), (b,c) \rangle$.

Factoring N by $NL[5]$ gives, N is an extension of $NL[5]$ by $N/NL[5]$. Furthermore, $q \cong N/NL[5]$ is of order 12 which is isomorphic to A_4 ; where the generators of q are: $q1 \sim (1,2)(3,4)$, $q2 \sim (1,3)(2,4)$, and $q3 \sim (2,3,4)$. We rename the generators of q as $q1 = d$, $q2 = e$, and $q3 = f$ which has the following presentation $\langle d, e, f \rangle = \langle d, e, f \mid d^2, e^2, f^3, (de)^2, ef^{-1}df, f^{-1}edfd \rangle$.

We equate every element of q with $N/NL[5]$ so we compute a sequence of transversals T of $NL[5]$ in N . We find the second element in the transversal $T[2]$ is equal to $q1$, $T[3]$ equal to $q2$, and $T[5]$ is equal to $q3$. Now, we conjugate every generator of $NL[5]$ by $T[2] \sim D$, $T[3] \sim E$, and $T[5] \sim F$. Thus, we compute $A^D = A$, $A^E = AC$, $A^F = AB$, $B^D = BC$, $B^E = BC$, $B^F = A$, $C^D = C$, $C^E = C$, and $C^F = C$. Therefore, the final presentation is $\langle a, b, c, d, e, f \mid a^2, b^2, c^2, (a,b), (a,c), (b,c), d^2, e^2, f^3, (de)^2, ef^{-1}df, f^{-1}edfd, a^d = a, a^e = ac, a^f = ab, b^d = bc, b^e = bc, b^f = a, c^d = c, c^e = c, c^f = c \rangle$. Thus, N is the semi-direct product $2^3:A_4$.

4.6 $2^4 : S_3$

We begin with a transitive group N of order 96 with generators $x \sim (3,9)(6,12)$, $y \sim (1, 10)(2, 5)(3, 12)(4, 7)(6, 9)(8, 11)$, $z \sim (1, 5, 9)(2, 6, 10)(3, 7, 11)(4, 8, 12)$, and $r \sim (1, 11)(2, 10)(3, 9)(4, 8)(5, 7)$. The normal lattice of N is

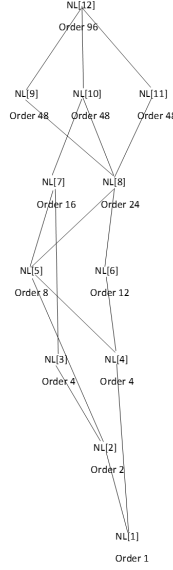


Figure 4.4: Normal Lattice of $2^4 : S_3$

The largest normal abelian subgroup of N is $NL[7]$ which has order 16. Thus, $NL[7]$ can be isomorphic to one of the following $\mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, or \mathbb{Z}_{16} . However, $NL[7]$ has four generators of order 2, such that the generators of $NL[7]$ are $A \sim (1, 10)(2, 5)(3, 12)(4, 7)(6, 9)(8, 11)$, $B \sim (2, 8)(5, 11)$, $C \sim (2, 8)(3, 9)(5, 11)(6, 12)$, and $D \sim (1, 7)(2, 8)(4, 10)(5, 11)$. Thus, $NL[7] \cong 2 \times 2 \times 2 \times 2 = 2^4$.

Since the only normal subgroup of order 2 is contained in $NL[7]$, implies N is not a direct product. Thus, N is a semi-direct product of $NL[7]$. A Presentation for $NL[7]$ is given by $\langle a, b, c, d \mid a^2, b^2, c^2, d^2, (a,b), (a,c), (a,d), (b,c), (b,d), (c,d) \rangle$. We factor N by $NL[7]$. Then, N is an extension of $NL[7]$ by $N/NL[7]$. Furthermore, $q \cong N/NL[7]$ has order 6 which is isomorphic to 2×3 . Generators of q are $G \sim (1, 2, 3)$ and $H \sim (2, 3)$. Note that q is not abelian and of order 6, thus S_3 is the only non-abelian group of order 6. So, $q \cong S_3$. A presentation for q is given by $\langle g, h \mid g^3, h^2, (gh)^2 \rangle$.

We compute a sequence of transversals T of $NL[7]$ in N . We obtain $T[2]$ is equal to $q3$ and $T[3]$ is equal to $q4$. Let $T[2] \sim G$ and $T[3] \sim H$. We now conjugate every generator of $NL[7]$ by G and H and compute $A^G = A$, $A^H = ABCD$, $B^G = BC$, $B^H = BD$, $C^G = CD$, $C^H = CD$, $D^G = C$, and $D^H = D$. Therefore, the presentation for N is given by $\langle a, b, c, d, g, h \mid a^2, b^2, c^2, d^2, (a,b), (a,c), (a,d), (b,c), (b,d), (c,d), g^3, h^2, (gh)^2, a^g = a, a^h = abcd, b^g = bc, b^h = bd, c^g = cd, c^h = cd, d^g = c, d^h = d \rangle$. Thus, N is the semi-direct product $2^4 : S_3$.

4.7 Mixed Extension $2^7 : \bullet 2^6$

We begin with a transitive group N of order 57,344 acting on a set of cardinality 28. N has generators $x \sim (1, 6, 12, 16, 20, 23, 27, 4, 8, 9, 13, 18, 22, 25)(2, 5, 11, 15, 19, 24, 28, 3, 7, 10, 14, 17, 21, 26)$ and $y \sim (1, 12, 17, 25, 6, 15, 23, 3, 10, 20, 28, 8, 14, 22)(2, 11, 18, 26, 5, 16, 24, 4, 9, 19, 27, 7, 13, 21)$. The Normal Lattice of N is

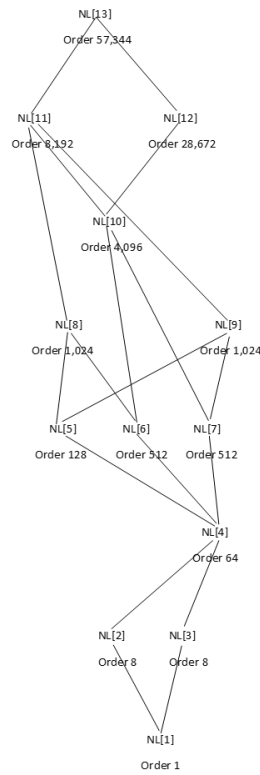


Figure 4.5: Normal Lattice of Mixed Extension $2^7 : \bullet 2^6$

The largest normal abelian subgroup of N is $NL[5]$ has order 128. The generators of $NL[5]$ are:

$$A = (1, 3)(2, 4)(5, 8)(6, 7)(9, 12)(10, 11)(13, 15)(14, 16)(17, 20)(18, 19) (21, 23)(22, 24) (25, 28)(26, 27),$$

$$B = (21, 22)(23, 24)(25, 26)(27, 28),$$

$$C = (17, 18)(19, 20)(21, 22)(23, 24),$$

$$D = (13, 14)(15, 16)(25, 26)(27, 28),$$

$$E = (1, 2)(3, 4)(5, 6)(7, 8)(17, 18)(19, 20)(25, 26)(27, 28),$$

$$F = (5, 6)(7, 8)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24), \text{ and}$$

$$G = (9, 10)(11, 12)(17, 18)(19, 20)(21, 22)(23, 24)(25, 26)(27, 28).$$

From the generators of $NL[5]$, $NL[5] \cong 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7$. A presentation of $NL[5]$ is $\langle a, b, c, d, e, f, g \mid a^2, b^2, c^2, d^2, e^2, f^2, g^2, (a,b), (a, c), (a,d), (a,e), (a,f), (a,g), (b,c), (b,d), (b,e), (b,f), (b,g), (c,d), (c,e), (c,f), (c,g), (e,f), (e,g) \rangle$.

Factoring N by $NL[5]$, N is an extension of $NL[5]$ by $N/NL[5]$. Furthermore, $N/NL[5] \cong q = \langle H, J \rangle$ of order 448. N is not a direct product of $NL[5]$ because N does not have a normal subgroup of order 448. Therefore, N is a semi-direct product of $NL[5]$, where $H = (2, 3, 5, 9, 16, 27, 41)(4, 7, 13, 23, 36, 17, 29)(6, 11, 20, 33, 10, 18, 30)(8, 15, 26, 14, 25, 12, 22)(19, 31, 44, 40, 35, 47, 58)(21, 34, 45, 59, 63, 46, 61)(24, 37, 50, 56, 64, 39, 32)(28, 42, 55, 51, 60, 54, 53)(38, 52, 62, 49, 43, 57, 48)$ and $J = (1, 2, 4, 8, 15, 23, 27)(3, 6, 12, 13, 24, 38, 53)(5, 10, 19, 32, 18, 9, 17)(7, 14, 20, 16, 28, 43, 58)(11, 21, 35, 48, 57, 50, 61)(22, 26, 40, 54, 41, 42, 56)(25, 39, 36, 49, 52, 29, 31)(30, 34, 46, 33, 37, 51, 47)(44, 45, 60, 62, 55, 63, 64)$. Thus, a presentation for $N/NL[5]$ is $\langle h, j \mid h^7, j^7, (hj^{-1}h)^2, (h^{-1}, j^{-1})^2, (h^{-1}j^{-3})^2 \rangle$.

We compute a sequence of transversals T of $NL[5]$ in N , such that $T[2] \sim H$ and $T[3] \sim J$. We compute $A^{T[2]} = ACDG$, $A^{T[3]} = ACEG$, $B^{T[2]} = BDEF$, $B^{T[3]} = BCE$, $C^{T[2]} = B$, $C^{T[3]} = BDEF$, $D^{T[2]} = CDEF$, $D^{T[3]} = BCDF$, $E^{T[2]} = EG$, $E^{T[3]} = FG$, $F^{T[2]} = G$, $F^{T[3]} = EF$, $G^{T[2]} = EF$, and $G^{T[3]} = E$. Therefore, the presentation for N is given by $\langle a, b, c, d, e, f, g, h, j \mid a^2, b^2, c^2, d^2, e^2, f^2, g^2, (a,b), (a,c), (a,d), (a,e), (a,f), (a,g), (b,c), (b,d), (b,e), (b,f), (b,g), (c,d), (c,e), (c,f), (c,g), (e,f), (e,g), h^7, j^7, (hj^{-1}h)^2, (h^{-1}, j^{-1})^2, (h^{-1}j^{-3})^2, a^h = acdg, a^j = aceg, b^h = bdef, b^j = bce, c^h = b, c^j = bdef, d^h = cdef, d^j = bcdf, e^h = eg, e^j = fg, f^h = g, f^j = ef, g^h = ef, g^j = e \rangle$ and has isomorphism type $2^7 : 2^6$.

We check if the presentation is isomorphic to N by using the following MAGMA code.

```
> M<a,b,c,d,e,f,g,h,j>:=Group<a,b,c,d,e,f,g,h,j|a^2,b^2,c^2,d^2,e^2,
f^2,g^2,(a,b),(a,c),(a,d),(a,e),(a,f),(a,g),
(b,c),(b,d),(b,e),(b,f),(b,g),(c,d),(c,e),(c,f),
(c,g),(e,f),(e,g),h^7,j^7,(h*j^-1*h)^2,(h^-1,j^-1)^2,
(h^-1*j^-3)^2,a^h=a*c*d*g,a^j=a*c*e*g,b^h=b*d*e*f,
b^j=b*c*e,c^h=b,c^j=b*d*e*f,d^h=c*d*e*f,d^j=b*c*d*f,
e^h=e*g,e^j=f*g,f^h=g,f^j=e*f,g^h=e*f,g^j=e>;
>
> f,M1,k:=CosetAction(M,sub<M|Id(M)>);>

> IsIsomorphic(M1,N);
false
```

It turns out that the above presentation did not give N . So, N must be a mixed extension. This implies some elements of $N/NL[5]$ can be written in terms of the elements of $NL[5]$. We proceed by checking the orders of the following elements of

$N/NL[5]$:

$h^7,$
 $j^7,$
 $(h * j^{-1} * h)^2,$
 $(h^{-1}, j^{-1})^2,$
 $(h^{-1} * j^{-3})^2$

In MAGMA we compute the following

```

> Order(q.1);
7
> Order(T[2]);
14
> Order(q.2); Order(T[3]);
7
14
> (q.1 * q.2^-1 * q.1)^2; (T[2] * T[3]^-1 * T[2])^2;
Id(q)
Id(N)
> (q.1^-1, q.2^-1)^2; (T[2]^-1, T[3]^-1)^2;
Id(q)
Id(N)
>
> (q.1^-1 * q.2^-3)^2; (T[2]^-1 * T[3]^-3)^2;
Id(q)
(5, 6)(7, 8)(13, 14)(15, 16)(17, 18)(19, 20)(25, 26)(27, 28)
>

```

Since, there is a homomorphism from N to $N/NL[5]$ we know $T[2], T[3] \in N$, but from above we see they each have order 14. We find $(NL[5]T[2])^7 = NL[5]$ which implies $NL[5]T[2]^7 = NL[5]$; which is $H^7 = \text{Id}(N)$, thus $T[2]^7 \in NL[5]$, similarly, $T[3]^7 \in NL[5]$ Thus, we can write the following in terms of $NL[5]$.

```

> T[2]^7;
(1, 4)(2, 3)(5, 7)(6, 8)(9, 12)(10, 11)(13, 16)(14, 15)
(17, 19)(18, 20)(21, 24)(22, 23)(25, 27)(26, 28)
> T[3]^7;

```

```

(1, 3)(2, 4)(5, 7)(6, 8)(9, 11)(10, 12)(13, 16)(14, 15)
(17, 20)(18, 19)(21, 24)(22, 23)(25, 28)(26, 27);
> (T[2]^-1 * T[3]^-3)^2
(5, 6)(7, 8)(13, 14)(15, 16)(17, 18)(19, 20)(25, 26)(27, 28)

```

we find

```

> T[2]^7 eq A*B*D*E;
true
> T[3]^7 eq A*B*F*G;
true
> (T[2]^-1 * T[3]^-3)^2 eq B*F;
true

```

Therefore, a final presentation for N is given by $\langle a, b, c, d, e, f, g, h, j$
 $| a^2, b^2, c^2, d^2, e^2, f^2, g^2, (a, b), (a, c), (a, d), (a, e), (a, f), (a, g), (b, c), (b, d), (b, e),$
 $(b, f), (b, g), (c, d), (c, e), (c, f), (c, g), (e, f), (e, g), h^7 = abde, j^7 = abfg, (hj^{-1}h)^2,$
 $(h^{-1}j^{-1})^2, (h^{-1}j^{-3})^2 = bf, a^h = acdg, a^j = aceg, b^h = bdef, b^j = bce, c^h = b,$
 $c^j = bdef, d^h = cdef, d^j = bcdf, e^h = eg, e^j = fg, f^h = g, f^j = ef, g^h = ef, g^j =$
 $e \rangle$ Thus, N is mixed extension $2^7 : \bullet 2^6$.

4.8 Mixed Extension $2^3 : \bullet 4$

We begin with a transitive group N of order 32 acting on a set of cardinality 8. N has generators $x \sim (2,6)(3,7)$ and $y \sim (1,2,3,4,5,6,7,8)$. The Normal Lattice of N is given below.

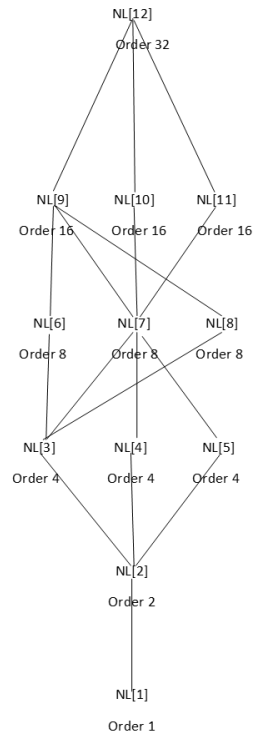


Figure 4.6: Normal Lattice of Mixed Extension $2^3 : \bullet 4$

The largest normal abelian subgroup of N is $NL[8]$ which has order 8. The generators of $NL[8]$ are $A = (1, 3)(2, 4)(5, 7)(6, 8)$, $B = (2, 6)(4, 8)$, and $C = (1, 5)(3, 7)$. From the generators of $NL[8]$, $NL[8] \cong 2 \times 2 \times 2 = 2^3$. Thus, a

presentation of $NL[8]$ is given by $\langle a, b, c \mid a^2, b^2, c^2, (a,b), (a, c), (b,c) \rangle$.

Now, factoring N by $NL[8]$, such that, N is an extension of $NL[8]$ by $N/NL[8]$. Furthermore, $N/NL[8] \cong q = \langle NL[8]D, NL[8]E \rangle$ of order 4. N is not a direct product of $NL[8]$ because N does not have a normal subgroup of order 4. Therefore, N is a semi-direct product of $NL[8]$, where $D = (1, 2)(3, 4)$ and $E = (1, 3, 2, 4)$. Thus, a presentation for $N/NL[8]$ is given by $\langle d, e \mid d^2, e^{-2}d \rangle$.

We compute a sequence of transversals T of $NL[8]$ in N , such that $T[2] \sim (2,6)(3,7)$ and $T[3] \sim (1,2,3,4,5,6,7,8)$, and $T[4] \sim (6,2)(7,3)$. We compute $A^{T[2]} = ABC$, $A^{T[3]} = AC$, $A^{T[4]} = AB$, $B^{T[2]} = B$, $B^{T[3]} = C$, $B^{T[4]} = C$, $C^{T[2]} = C$, $C^{T[3]} = B$, and $C^{T[4]} = B$. Therefore, the presentation for N is given by $\langle a, b, c, d, e \mid a^2, b^2, c^2, d^2, e^{-2}d, (a,b), (a, c), (b,c), a^d = abc, a^e = ac, a^{e^{-1}} = ab, b^d = b, b^e = c, b^{e^{-1}} = c, c^d = c, c^e = b, c^{e^{-1}} = b \rangle$ which has isomorphism type $2^3 : 4$.

We check if the presentation is isomorphic to N by using the following MAGMA code.

```
> H<a,b,c,d,e>:=Group<a,b,c,d,e|a^2,b^2,c^2,(a,b),(a,c),(b,c),
d^2,e^-2*d, a^d=a*b*c, a^e=a*c, a^(e^-1)=a*b, b^d=b,b^e=c,b^(e^-1)=c,
c^d=c, c^e=b, c^(e^-1)=b>;
> #H;
32
> f,H1,k:=CosetAction(H,sub<H|Id(H)>);
> IsIsomorphic(N,H1);
false
```

It turns out that the above presentation did not give N . So, N must be a mixed extension. This implies some elements of $N/NL[8]$ can be written in terms of the

elements of $NL[8]$. We proceed by checking the orders of the following elements of $N/NL[8]$:

d^2 and e^{-2*d}

In MAGMA we compute the following

```
> T[2];
(2, 6)(3, 7)
> T[3];
(1, 2, 3, 4, 5, 6, 7, 8)
>
> T[4];
(1, 8, 7, 6, 5, 4, 3, 2)
>
>
> T[2];
(2, 6)(3, 7)
>
> Order(T[2]);
2
>
> Order(q.1);
2
>
> Order(T[3]);
8
> Order(q.2);
4
>
> Order(T[4]);
8
> Order(q.-2);
4
```

Since, there is a homomorphism from N to $N/NL[8]$ we know $T[2]$, $T[3]$, and $T[4] \in N$, but from above we see they each have order 2, 8, and 8 respectively. Thus,

we can write the following in terms of the elements of NL[8].

```
> T[3]^(-2)*T[2];
(1, 3)(2, 8)(4, 6)(5, 7)
>
> Order(T[3]^(-2)*T[2]);
2
> for i,j,m in [0..1] do if (T[3]^(-2) * T[2]) eq A^i*B^j*C^m then
for|if>   i,j,m; end if; end for;
1 1 0

> T[3]^(-2)*T[2] eq A*B;
true
```

and use MAGMA to verify if the calculations are correct.

```
> H<a,b,c,d,e>:=Group<a,b,c,d,e|a^2,b^2,c^2,(a,b),(a,c),(b,c),d^2,
a^d=a*b*c, a^e=a*c, a^(e^-1)=a*b, b^d=b,b^e=c,b^(e^-1)=c,
c^d=c, c^e=b, c^(e^-1)=b,e^-2*d=a*b>;
> #H;
32
> f,H1,k:=CosetAction(H,sub<H|Id(H)>);
> IsIsomorphic(N,H1);
true Mapping from: GrpPerm: N to GrpPerm: H1
Composition of Mapping from: GrpPerm: N to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: H1
```

The final presentation for N is given by $\langle a, b, c, d, e \mid a^2, b^2, c^2, d^2, (a,b), (a,c), (b,c), a^d = abc, a^e = ac, a^{e^{-1}} = ab, b^d = b, b^e = c, b^{e^{-1}} = c, c^d = c, c^e = b, c^{e^{-1}} = b, e^{-2}d = ab \rangle$. Thus, N is mixed extension $2^3 :^\bullet 4$.

4.8.1 The MAGMA Command FPG

It should be pointed out that the Magma FPG (Finitely Presented Group Presentation) command does not always give a satisfactory presentation even in the obvious cases such as the one discussed below. Thus, in each application of FPG, we should simplify the presentation before using it. We can solve the extension problem of the previous example by reducing the number of generators of $q \cong N/NL[8]$, and adjusting the given presentation $\langle a, b, c, d, e \mid a^2, b^2, c^2, d^2, (a,b), (a, c), (b,c), a^d = abc, a^e = ac, a^{e^{-1}} = ab, b^d = b, b^e = c, b^{e^{-1}} = c, c^d = c, c^e = b, c^{e^{-1}} = b, e^{-2}d = ab \rangle$ for N .

Previously, we have factored N by $NL[8]$, where N is an extension of $NL[8]$ by $N/NL[8]$. Furthermore, $N/NL[8] \cong q = \langle NL[8]D, NL[8]E \rangle$ of order 4; where $D = (1,2)(3,4)$ and $E = (1,3,2,4)$. Thus, a presentation for $N/NL[8]$ is given by $\langle d, e \mid d^2, e^{-2}d \rangle$. However, we notice the generators of $N/NL[8] \cong q$ are $NL[8]D = NL[8](1,2)(3,4)$ has order 2 and $NL[8]E = NL[8](1,3,2,4)$ has order 4. Since, $N/NL[8]$ has order 4 we have that it contains elements $NL[8]$, $NL[8]x$, $NL[8]x^2$, $NL[8]x^3$. Thus, $(NL[8]E)^2$ is of order 2, which implies $NL[8]D$ is redundant; which means q can solely be generated by $NL[8]E$. Thus, a presentation for q is given by $\langle d \mid d^4 \rangle$ (for technicality purposes we denote E by d).

We compute a sequence of transversals T of $NL[8]$ in N , such that $T[3] \sim (1,2,3,4,5,6,7,8)$. We compute $A^{T[3]} = AC$, $B^{T[3]} = C$, and $C^{T[3]} = B$. Therefore, the presentation for N is given by $\langle a, b, c, d \mid a^2, b^2, c^2, d^4, (a,b), (a,c), (b,c), a^d = ac, b^d = c, c^d = b \rangle$ which has isomorphism type $2^3 : 4$.

We check if the presentation is isomorphic to N by using the following MAGMA code.

```
> H<a,b,c,d>:=Group<a,b,c,d|a^2,b^2,c^2,(a,b),(a,c),(b,c),
a^d=a*c, b^d=c, c^d=b>;
> #H;
32
> f,H1,k:=CosetAction(H,sub<H|Id(H)>);
> IsIsomorphic(N,H1);
false
```

It turns out that the above presentation did not give N . So, N must be a mixed extension. This implies some elements of $N/NL[8]$ can be written in terms of $NL[8]$.

We proceed by checking the orders of the following elements of $N/NL[8]$:

d^4

In MAGMA we compute the following

```
> T[3];
(1, 2, 3, 4, 5, 6, 7, 8)
>
> Order(T[3]);
8
> Order(q.2);
4
```

Since, there is a homomorphism from N to $N/NL[8]$ we know $T[3] \in N$, but from above we see they each have order 8 and 4, respectively. Thus, $(NL[8]T[3])^4 = NL[8]$ implies $NL[8]T[3]^4 = NL[8]$; which is $E^4 = \text{Id}(N)$. Thus, $T[3]^4 \in NL[8]$ and write the following term as an element of $NL[8]$.

```
> T[3]^4
(1, 5)(2, 6)(3, 7)(4, 8)
>
> T[3]^4 eq B*C;
true
```

Thus, we confirm the calculations are correct in MAGMA

```

H<a,b,c,d>:=Group<a,b,c,d|a^2,b^2,c^2,(a,b),(a,c),(b,c), a^d=a*c, b\
^d=c,c^d=b,d^4=b*c >;
>
> #H;
32
> f,H1,k:=CosetAction(H,sub<H|Id(H)>);
> IsIsomorphic(N,H1);
true Mapping from: GrpPerm: N to GrpPerm: H1
Composition of Mapping from: GrpPerm: N to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: H1

```

Therefore, we have reduced the number of generators of N and adjusted the final presentation for N as $\langle a, b, c, d \mid a^2, b^2, c^2, d^4=bc, (a,b), (a, c), (b,c), a^d = ac, b^d = c, c^d = b \rangle$. Thus, N is mixed extension $2^3 :^\bullet 4$.

Chapter 5

Double Cosets

5.1 Preliminaries

Definition 5.1.1. (**right coset**) If S is a subgroup of G and if $t \in G$, then a **right coset** of $S \in G$ is the subset of G : $St = \{st:s \in S\}$ (a **left coset** is $tS = \{ts:s \in S\}$). One calls t a **representative** of St (and also tS). [Rot95]

Theorem 5.1.2. If $S \leq G$, then any two right (or any two left)cosets of S in G are either identical or disjoint. [Rot95]

Theorem 5.1.3. If $S \leq G$, then the number right cosets of S in G is equal to the number of left cosets of S in G . [Rot95]

Definition 5.1.4. (**index**) If $S \leq G$, then the **index** of S in G , denoted $[G:S]$, is the number of right cosets of S in G . [Rot95]

Definition 5.1.5. (conjugate) If $x \in G$, then a **conjugate** of x in G is an element of the form axa^{-1} for some $a \in G$. [Rot95]

Definition 5.1.6. (double coset) If S and T are subgroups of G , then a **double coset** is a subset of G of the form SgT , where $g \in G$. [Rot95]

Definition 5.1.7. (G-set) If X is a set and G is a group, then X is a **G-set** if there is a function $\alpha: G \times X \longrightarrow X$ (called an **action**), denoted by $\alpha: (g, x) \mapsto gx$, such that:

- (i) $1x = x$ for all $x \in X$; and
- (ii) $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$. [Rot95]

Definition 5.1.8. (acts) G **acts** on X , if $|X| = n$, then n is called the **degree** of the G -set X . [Rot95]

Definition 5.1.9. (G-orbit) If X is a G -set and $x \in X$, then the **G-orbit** of x is $\vartheta(x) = \{gx : g \in G\} \subset X$, ($\vartheta(x)$ denoted Gx). [Rot95]

Definition 5.1.10. (stabilizer) If X is a G -set and $x \in X$, then the **stabilizer** of x , denoted by G_x , is the subgroup $G_x = \{g \in G : gx = x\} \leq G$. [Rot95]

Theorem 5.1.11. If X is a G -set and $x \in X$, then $|\vartheta(x)| = [G : G_x]$. [Rot95]

Corollary 5.1.12. *If a finite group G acts on a set X , then the number of elements in any orbit is a divisor of $|G|$. [Rot95]*

Corollary 5.1.13. *(i) If G is a finite group and $x \in G$, then the number of conjugates of $x \in G$ is $[G : C_G(x)]$ (C_G , is **centralizer**). (ii) If G is a finite group and $H \leq G$, then the number of conjugates of $H \in G$ is $[G : N_G(H)]$ (N_G , is **normalizer**). [Rot95]*

Definition 5.1.14. (transitive) *A G -set X is **transitive** if it has only one orbit; that is for every $x, y \in X$, there exists $\sigma \in G$ with $y = \sigma x$. [Rot95]*

Definition 5.1.15. (Double Coset Algorithm) *Perform the double coset enumeration of group G over transitive group N , where double cosets take the form $NwN = \{Nwn \mid n \in N\} = \{Nw^n \mid n \in N\}$.*

- (i) Compute the point-stabilizer N^w and coset stabilizer of each double coset.*
- (ii) Compute the number of right cosets by using the formula $\frac{|N|}{|N^{(w)}|}$, where $N^{(w)} = \{n \in N \mid Nw^n = Nw\}$ is the coset stabilizer of the right coset.*
- (iii) For each double coset NwN , compute the orbits of $N^{(w)}$. It suffices to determine the double coset of Nwt_i for a single representative of each orbit. Note, $N^{(w)} \geq N^w$ is always true.*
- (iv) Determine which double coset each coset representative Nwt_i belongs to, (repeat the process until closed by coset multiplication) .*

5.2 DCE $3^3 : 2^4$ over $3^2 : 2^2$

A symmetric presentation of the progenitor $2^{*6} : (3^2 : 2^2)$ is $\langle x, y, z, t \mid x^3, y^2, z^2, (x^{-1}y)^2, (yz)^2, x^{-1}z x^{-1}zxzxz, t^2, (t, xyzxz), (t, x) \rangle$ where $N \cong (3^2 : 2^2)$ is of order 36, and $x \sim (2, 4, 6)$, $y \sim (1, 5)(2, 4)$, $z \sim (1, 4)(2, 5)(3, 6)$, and $t \sim t_1$.

We prove the above progenitor factored by $(zxt)^4$ and $(t_1 t_1^{zy})^2$ is isomorphic to $3^3 : 2^4$. We expand our relations as follows:

Relation $(zxt)^4 = 1$

$$(zxt)^4 = 1$$

$$\implies ((1, 6, 3, 2, 5, 4)t_1)^4$$

$$\implies (\pi t_1)^4 \text{ (let } (1, 6, 3, 2, 5, 4) = \pi)$$

$$\implies \pi t_1 \pi t_1 \pi t_1 \pi t_1$$

$$\implies \pi \pi \pi^{-1} t_1 \pi t_1 \pi t_1 \pi t_1$$

$$\implies \pi^2 \pi \pi^{-1} t_1 \pi t_1 \pi t_1 \pi t_1$$

$$\implies \pi^3 \pi \pi^{-1} t_1 \pi^2 t_1 \pi t_1 \pi t_1$$

$$\implies \pi^4 t_1 \pi^3 t_1 \pi^2 t_1 \pi t_1$$

$$\implies (1, 5, 3)(2, 6, 4)t_2 t_3 t_6 t_1 = 1.$$

Relation $(t_1 t_1^{zy})^2 = 1$

$$\implies (t_1 t_1^{(1,2)(3,6)(4,5)})^2$$

$$\implies (t_1 t_2)^2$$

$$\implies t_1 t_2 t_1 t_2 = 1.$$

Thus, our homomorphic image is $G \cong (2^4 : 3^3) \cong \frac{2^{*6} : (2^2 : 3^2)}{(zxt)^4 = 1, (tt^{zy})^2 = 1}$.

Now we proceed with the Double Coset Enumeration of G over N .

The First Double Coset $NeN = [*]$

Let NeN be denoted by $[*]$. $NeN = \{Nen \mid n \in N\} = \{Nne^n \mid n \in N\} = \{Ne^n \mid n \in N\} = \{N\}$. N is the only single coset contained in the double coset $[*]$. The coset representative of the double coset $[*]$ is N . The coset stabilizer in N of the coset N is N . Thus $\frac{|N|}{|N|} = \frac{36}{36} = 1$.

Since N is transitive on $\{1,2,3,4,5,6\}$, it has the single orbit $\{1,2,3,4,5,6\}$. We choose 1 to be the representative form the orbit $\{1,2,3,4,5,6\}$, and form the new double coset, $Nt_1N = \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6\}$. We note that all 6 symmetric generators from the orbit extend to Nt_1N .

The Second Double Coset $Nt_1N = [1]$

The number of cosets contained in this double coset is given by $\frac{|N|}{|N^{(1)}|}$, where $N^{(1)}$ is the coset stabilizer of the coset Nt_1 which is always greater than or equal to the point stabilizer of 1 in N , denoted by N^1 . $N^{(1)} \geq N^1 = \{\text{Id}(N), (3, 5)(4, 6), (2, 4)(3, 5), (2, 4, 6), (2, 6, 4), (2, 6)(3, 5)\} = \langle (2, 4, 6), (2, 4)(3, 5) \rangle \cong S_3$ and is visibly of order 6. Therefore, the number of single cosets in the double coset Nt_1N is $\frac{|N|}{|N^{(1)}|} = \frac{36}{6} = 6$.

The orbits of $N^{(1)}$ on $\{1,2,3,4,5,6\}$ are $\{1\}$, $\{2,4,6\}$, and $\{3,5\}$. We choose one 1, 2, and 3 to be representatives form each orbit respectively and form the cosets Nt_1t_1 , Nt_1t_2 , and Nt_1t_3 . We determine which double cosets each coset representative Nt_1t_1 , Nt_1t_2 , and Nt_1t_3 belongs to. It is easy to see that $Nt_1t_1 \in [*]$, since $Nt_1t_1 = Nt_1^2 = N$. Therefore, one symmetric generator goes back to $[*]$.

Now, $Nt_1t_2 \in Nt_1t_2N = [12]$ so, 3 symmetric generators extend to [12]; and $Nt_1t_3 \in Nt_1t_3N = [13]$ so, 2 symmetric generators extend to [13].

Third Double Coset $Nt_1t_2N = [12]$

Now, $N^{12} = \langle (3,5)(4,6) \rangle$. However, by the relation $t_1t_2t_1t_2 = 1$ gives $12 = 21$, which implies $Nt_1t_2 = Nt_2t_1$. Thus, $12 \sim 21$. Now, $N(t_1t_2)^{(1,2)(3,6)(4,5)} = Nt_2t_1 = Nt_1t_2$. Therefore, $(1,2)(3,6)(4,5) \in N^{(12)}$; such that $N^{(12)} \geq \langle N^{12}, (1,2)(3,6)(4,5) \rangle$. Also, when we conjugate the relation $(1,5,3)(2,6,4)t_2t_3t_6t_1 = 1$ by $(2, 4, 6)$ we obtain $(1,5,3)(4,2,6)t_4t_3 = t_1t_2$. Thus, $Nt_1t_2 = Nt_4t_3$ implies $12 \sim 43$.

Now, $N(t_1t_2)^{(1,4,5,2,3,6)} = Nt_4t_3 = Nt_1t_2$ implies $(1, 4, 5, 2, 3, 6) \in N^{(12)}$. Thus, $N^{(12)} \geq \{N^{12}, (1,2)(3,6)(4,5), (1, 4, 5, 2, 3, 6)\} = \langle (3,5)(4,6), (1, 4, 5, 2, 3, 6) \rangle \cong 6:2$ and $[1, 2]^{N^{(12)}}$ gives $12 \sim 56 \sim 21 \sim 65 \sim 43 \sim 34$. Therefore, the number of single cosets in the double coset Nt_1t_2N is $\frac{|N|}{|N^{(12)}|} = \frac{36}{12} = 3$. The orbit of $N^{(12)}$ on $\{1,2,3,4,5,6\}$ is $\{1,2,3,4,5,6\}$. We take 2 to be the representative from the orbit and form the coset $Nt_1t_2t_2$. However, $Nt_1t_2t_2 = Nt_1t_2^2 = Nt_1 \in [1]$ so 6 symmetric generators go back to [1].

The Fourth Double Coset $Nt_1t_3N = [13]$

Now, $N^{13} = \{e, (2,4,6), (2,6,4)\} = \langle (2,4,6) \rangle$.

Lemma 13 = 26

Proof:

The relation $(1, 5, 3)(2, 6, 4)t_2t_3t_6t_1 = 1$

conjugated by $(1,3)(2,4)$ gives $(1,3,5)(2,4,6)t_4t_1t_3t_6 = 1$

$$\implies (1,3,5)(2,4,6)t_4t_1t_3 = t_6$$

$$\implies t_4t_1t_3 = (1,5,3)(2,6,4)t_6$$

$$\implies t_1t_3 = t_4(1,5,3)(2,6,4)t_6$$

$$\implies t_1t_3 = (1,5,3)(2,6,4)t_2t_6$$

Thus, $Nt_1t_3 = Nt_2t_6$ and $13 \sim 26$. \square

Now, $N(t_1t_3)^{(1,2,3,6,5,4)} = Nt_2t_6 = Nt_1t_3$ implies $(1,2,3,6,5,4) \in N^{(13)}$.

So, $N^{(13)} \geq \langle N^{13}, (1,2,3,6,5,4) \rangle$ and $[1,3]^{N^{(13)}}$ gives $13 \sim 26 \sim 42 \sim 64 \sim 51 \sim 35$. Therefore, the number of single cosets in the double coset Nt_1t_3N is

$$\frac{|N|}{|N^{(13)}|} = \frac{36}{18} = 2.$$

The orbit of $N^{(13)}$ on $\{1,2,3,4,5,6\}$ is $\{1,2,3,4,5,6\}$. We take 3 to be the representative for this orbit and form the coset $Nt_1t_3t_3$. However, $Nt_1t_3t_3 = Nt_1t_3^2 = Nt_1 \in [1]$. So 6 symmetric generators go back to $[1]$. Thus, we have completed the double coset enumeration.

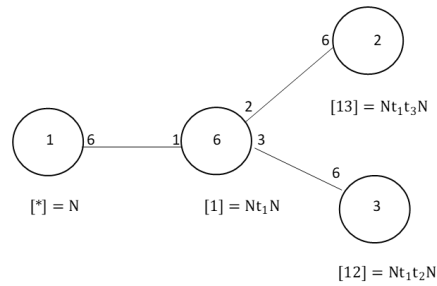


Figure 5.1: Cayley Graph of $3^3 : 2^4$ over $3^2 : 2^2$

5.2.1 Proof of $3^3 : 2^4 \cong \frac{2^{*6}:(3^2:2^2)}{(zxt)^4=1, (tt^{zy})^2=1}$

The double coset decomposition is as follows:

$$G \leq N \cup Nt_1N \cup Nt_1t_2N \cup Nt_1t_3N = (1+6+2+3) \times |N| = 12 \times 36 = 432.$$

Let $G \cong \frac{2^{*6}:(3^2:2^2)}{(zxt)^4=1, (tt^{zy})^2=1}$. We begin by finding the permutation representation of G by computing the action of G on the cosets we found above. We will show $|G| = 432$. From the coset decomposition above we have that $|G| \leq 432$, this implies the maximum order of G is 432. We label the right cosets by numbers 1,...,12 as shown below.

Cosets of NnN :

1 N

Cosets of Nt_1N :

2 Nt_1

3 Nt_2

4 Nt_3

5 Nt_4

6 Nt_5

7 Nt_6

Cosets of Nt_1t_2N :

8 Nt_1t_2

9 Nt_1t_4

10 Nt_1t_6

Cosets of Nt_1t_3N :

11 Nt_1t_3

12 Nt_2t_4

We let x , y , and z be the generators of the group $3^2 : 2^2$ of order 36, and t a symmetric generator. The given relation helps us compute the action on the 12 right cosets, such that conjugating the relation by all the elements of $3^2 : 2^2$ will tell us which right cosets are equal.

$$\begin{aligned}
Nt_1t_2 &= Nt_5t_6 = Nt_2t_1 = Nt_6t_5 = Nt_3t_4 = Nt_4t_3 \\
Nt_1t_4 &= Nt_5t_2 = Nt_4t_1 = Nt_2t_5 = Nt_3t_6 = Nt_6t_3 \\
Nt_1t_6 &= Nt_5t_4 = Nt_4t_5 = Nt_6t_1 = Nt_3t_2 = Nt_2t_3 \\
Nt_1t_3 &= Nt_2t_6 = Nt_4t_2 = Nt_6t_4 = Nt_3t_5 = Nt_5t_1 \\
Nt_2t_4 &= Nt_5t_3 = Nt_4t_6 = Nt_6t_2 = Nt_1t_5 = Nt_3t_1
\end{aligned}$$

We now compute the action of x , y and z on the 12 right cosets.

Table 5.1: Action of $x \sim (2,4,6)$

1	2	3	4	5	6	7	8	9	10	11	12
N	Nt_1	Nt_2	Nt_3	Nt_4	Nt_5	Nt_6	Nt_1t_2	Nt_1t_4	Nt_1t_6	Nt_1t_3	Nt_2t_4
\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow
N	Nt_1	Nt_4	Nt_3	Nt_6	Nt_5	Nt_2	Nt_1t_4	Nt_1t_6	Nt_1t_2	Nt_1t_3	Nt_2t_4
1	2	5	4	7	6	3	9	10	8	11	12

Table 5.2: Action of $y \sim (1,5)(2,4)$

1	2	3	4	5	6	7	8	9	10	11	12
N	Nt_1	Nt_2	Nt_3	Nt_4	Nt_5	Nt_6	Nt_1t_2	Nt_1t_4	Nt_1t_6	Nt_1t_3	Nt_2t_4
\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow
N	Nt_5	Nt_4	Nt_3	Nt_2	Nt_1	Nt_6	Nt_1t_6	Nt_1t_4	Nt_1t_2	Nt_2t_4	Nt_1t_3
1	6	5	4	3	2	7	10	9	8	12	11

Table 5.3: Action of $z \sim (1,4)(2,5)(3,6)$

1	2	3	4	5	6	7	8	9	10	11	12
N	Nt_1	Nt_2	Nt_3	Nt_4	Nt_5	Nt_6	Nt_1t_2	Nt_1t_4	Nt_1t_6	Nt_1t_3	Nt_2t_4
\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow
N	Nt_4	Nt_5	Nt_6	Nt_1	Nt_2	Nt_3	Nt_1t_6	Nt_1t_4	Nt_1t_2	Nt_2t_4	Nt_1t_3
1	5	6	7	2	3	4	10	9	8	12	11

Table 5.4: Action of $t \sim t_1$

1	2	3	4	5	6
N	Nt_1	Nt_2	Nt_3	Nt_4	Nt_5
\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow
Nt_1	N	Nt_1t_2	Nt_2t_4	Nt_1t_4	Nt_1t_3
2	1	8	12	9	11

7	8	9	10	11	12
Nt_6	Nt_1t_2	Nt_1t_4	Nt_1t_6	Nt_1t_3	Nt_2t_4
\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow
Nt_1t_6	Nt_2	Nt_4	Nt_6	Nt_5	Nt_3
10	3	5	7	6	4

From the labeling we obtain the permutations $f(x)=(3,5,7)(8,9,10)$, $f(y)=(2,6)(3,5)(8,10)(11,12)$, $f(z)=(2,5)(3,6)(4,7)(8,10)(11,12)$, and $f(t)=(1,2)(3,8)(4,12)(5,9)(6,11)(7,10)$.

We now have $S_{12} \geq \langle f(x), f(y), f(z), f(t) \rangle$, this tells us there is a homomorphism $\phi : G \longrightarrow S_{12}$. Thus, $\phi(G) = \langle f(x), f(y), f(z), f(t) \rangle = \langle x, y, z, t \rangle$. However, by the First Isomorphism Theorem we have $G/\ker(\phi) \cong \text{Im}(\phi) = \langle f(x), f(y), f(z), f(t) \rangle$. Therefore, $|G/\ker(\phi)| \cong |\langle f(x), f(y), f(z), f(t) \rangle| = |\phi(G)|$, and $\frac{|G|}{|\ker(\phi)|} = |\langle f(x), f(y), f(z), f(t) \rangle| = |\phi(G)|$. This implies $|G| = |\ker(\phi)| |\langle f(x), f(y), f(z), f(t) \rangle| = |\phi(G)|$ and $|G| \geq 432$, but from our cayley diagram above we have $|G| \leq 432$, therefore $|G|=432$ and $\ker(\phi)=1$. Thus, $G \cong \langle f(x), f(y), f(z), f(t) \rangle = \phi(G) \cong 3^3 : 2^4$. \square

Chapter 6

Double Cosets $PGL(2,11)$ and $L(2,11)$

6.1 DCE $PGL(2,11)$ over A_5

A symmetric presentation of the progenitor $2^6:A_5$ is $\langle x, y, t \mid x^5, y^2, (yx^{-1})^3, t^2, (t, (yx^2)^2), (t, xyx^{-1}) \rangle$. $N \leq S_6$, such that $N \cong A_5$; where $x \sim (1,2,3,4,6)$, $y \sim (1,4)(5,6)$, and $t \sim t_1$. We prove the above progenitor is factored by the relation $(x^2t)^4$ gives $PGL(2, 11)$.

We expand the relation as follows: $(x^2t)^4=1$

$$\implies x^2t_1x^2t_1x^2t_1x^2t_1=1$$

$$\implies (x^2)^4t_1^{(x^2)^3}t_1^{(x^2)^2}t_1^{(x^2)}t_1=1$$

$$\implies (1, 4, 2, 6, 3)t_2t_6t_3t_1=1.$$

This gives the finite homomorphic image of G , where $G \cong \frac{2^6:N}{(x^2t)^4}$. We proceed to find the index of N in G , by the double coset enumeration of G over N .

The First Double Coset $NeN = [*]$

Let NeN be denoted by $[*]$. $NeN = \{ Nen \mid n \in N \} = \{ Nne^n \mid n \in N \} = Ne^n = N$. Therefore, N is the only single coset contained in this double coset, and the number of single cosets in $[*]$ is $NeN = \frac{|N|}{|N|} = \frac{60}{60} = 1$. Now, N is transitive on $\{1,2,3,4,5,6\}$ so it has the single orbit $\{1,2,3,4,5,6\}$.

We choose 1 to be the representative from this orbit, and form the coset representative Nt_1 . We know only N is contained in $[*]$, so N is coset representative of $[*]$. Now $Nt_1 \in Nt_1N$, which is a new double coset, so 6 symmetric generators extend to this new double coset $Nt_1N = \{ N(t_1)^n \mid n \in N \} = \{ Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6 \} = [1]$.

The Second Double Coset $Nt_1N=[1]$

We find the point stabilizer of 1, denoted by N^1 , that is we find all elements of N which fix 1. $N^1 = \langle (2,3,5,4,6), (3,6)(4,5) \rangle$ and $|N^1| = 10$. The coset representative of 1, denoted $N^{(1)}$, contains the same elements as N^1 , thus $N^{(1)} = N^1$. The number of single cosets in Nt_1N is $\frac{|N|}{|N^{(1)}|} = \frac{60}{10} = 6$.

The orbits of $N^{(1)}$ on $\{1,2,3,4,5,6\}$ are $\{1\}$ and $\{2,3,4,5,6\}$. We choose 1 and 3 to be representatives, from each orbit and form the cosets representatives Nt_1t_1 and Nt_1t_3 . However, $Nt_1t_1 = Nt_1^2 = N$ thus, $Nt_1t_1 \in [*]$, such that 1 symmetric generator goes back to $[*]$. $Nt_1t_3 \in Nt_1t_3N=[13]$ is a new double coset, therefore 5 symmetric generators extend to $[13]$.

The Third Double Coset $Nt_1t_3N=[13]$

We find the point stabilizer of 1 and 3 in N , denoted by N^{13} , that is we find all elements of N which fix 1 and 3, is given by $N^{13} = \langle (2,5)(4,6) \rangle$ and is of order 2. So $N^{(13)} \geq N^{13} = \langle (2,5)(4,6) \rangle$.

Lemma: $13=54$

Proof:

by relation $Nt_1t_3t_6t_2=1$

relation can be re-written as $Nt_1t_3 = Nt_2t_6$ ($13=26$).

conjugating relation $Nt_1t_3 = Nt_2t_6$ by $(2,5)(4,6)$ gives $13=54$

Thus, $Nt_1t_3=Nt_5t_4$. \square

Now, $N(t_1t_3)^{(1,5)(3,4)} = Nt_5t_4 = Nt_1t_3$. So, $(1, 5)(3, 4) \in N^{(13)}$. Thus $N^{(13)} \geq \langle N^{13}, (1, 5)(3, 4) \rangle = \langle (2,5)(4,6), (1, 5)(3, 4) \rangle \cong S_3$. The orbits of $N^{(13)}$ on $\{1,2,3,4,5,6\}$ are $\{1,5,2\}$, and $\{3,4,6\}$. We choose a representative from each orbit and form the cosets $Nt_1t_3t_1$ and $Nt_1t_3t_3$. However, $Nt_1t_3t_3 = Nt_1t_3^2 = Nt_1 \in [1]$, so 3 symmetric generators go back to $[1]$. $Nt_1t_3t_1 \in Nt_1t_3t_1 = [131]$, therefore 3 symmetric generators extend to $[131]$.

The Fourth Double Coset $Nt_1t_3t_1N=[131]$

The point stabilizer $N^{131} = N^{13} = \langle (2, 5)(4, 6) \rangle$. The coset stabilizer $N^{(131)} \geq N^{13}$.

Lemma: $131=313$

Proof:

$$13=(1,4,2,6,3)26 \text{ (multiply by 1 on RHS)}$$

$$\implies 131=(1,4,2,6,3)\underline{261}$$

$$(\text{conjugate R1 by } (1,6,3)(2,4,5) \implies (6,5,4,3,1)43=61)$$

$$\implies 131=(1,4,2,6,3)2\underline{(6,5,4,3,1)43}$$

$$\implies 131=(1,4,2,6,3)(6,5,4,3,1)243$$

$$\implies 131=(1,3,6)(2,5,4)\underline{243}$$

$$\text{conjugate R1 by } (1,3)(4,6) \implies (3,6,2,4,1)24=31 \text{ and take inverse}$$

$$\implies 131=(1,3,6)(2,5,4)\underline{(1,4,2,6,3)313}$$

$$\implies 131=(2,5)(4,6)313$$

Thus, $Nt_1t_3t_1=Nt_3t_1t_3$. \square

Now $N(t_1t_3t_1)^{(1,3)(2,5)} = Nt_3t_1t_3 = Nt_1t_3t_1$. So, $(1,3)(2,5) \in N^{(131)}$.

Thus $N^{(131)} \geq N^{131}, (1,3)(2,5) >$.

Lemma: $131=525$

Proof:

$$131=(1,4,2,6,3)261 \text{ (multiply relation on RHS by 1)}$$

$$\implies 131=(1,4,2,6,3)\underline{261}$$

$$\text{conjugate R1 by } (1,6,5,4,3) \text{ gives } (6,3,2,5,1)25=61$$

$$\implies 131=(1,4,2,6,3)2\underline{(6,3,2,5,1)25}$$

$$\implies 131=(1,4,2,6,3)(6,3,2,5,1)525$$

$$\implies 131=(1,4,5)(2,3,6)525$$

Thus, $Nt_1t_3t_1=Nt_5t_2t_5$. \square

Now, $N(t_1t_3t_1)^{(1,5,6)(2,4,3)} = Nt_5t_2t_5 = Nt_1t_3t_1$. So $(1, 5, 6)(2, 4, 3) \in N^{(131)}$. Thus $N^{(131)} \geq < N^{131}, (1, 3)(2, 5), (1, 5, 6)(2, 4, 3) > \cong 2^2 \times 3$. Therefore, $N^{(131)}$ is of order 12. The number of single cosets in $Nt_1t_3t_1N$ is $\frac{|N|}{|N^{(131)}|} = \frac{60}{12} = 5$.

The orbit of $N^{(131)}$ on $\{1,2,3,4,5,6\}$ is the single orbit $\{1,2,3,4,5,6\}$. We take 1 to be the representative from this orbit and form the coset $Nt_1t_3t_1t_1$. However, $Nt_1t_3t_1t_1 = Nt_1t_3t_1^2 = Nt_1t_3 = Nt_1t_3 \in [13]$, so 6 symmetric generators go back to $[13]$. The set of cosets is closed under right multiplication by 6 symmetric generators.

Thus, $G \leq N \cup Nt_1N \cup Nt_1t_3N \cup Nt_1t_3t_1N = (1+6+10+5) \times |N| = (1+6+10+5) \times |60| = 1320$.

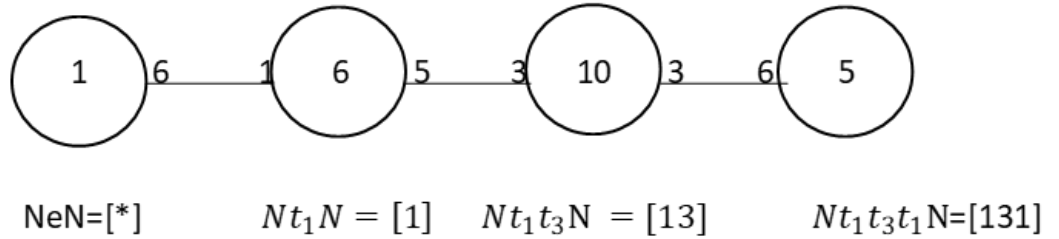


Figure 6.1: Cayley Graph of $PGL(2,11)$ over A_5

6.2 DCE L(2,11) over A_5

A symmetric presentation of the progenitor $2^{20} : A_5 \cong \langle x, y, t \mid x^5, y^3, (x^{-1}y^{-1})^2, x^{-1}y^{-3}xy^{-1}xy^3x^{-1}y, t^2, (t, xy^2x^{-2}y) \rangle$. $N \cong A_5$ of order 60 where $x \sim (1, 2, 6, 12, 4)(3, 9, 15, 16, 10)(5, 13, 8, 14, 7)(11, 17, 19, 20, 18)$, $y \sim (1, 3, 5)(2, 7, 8)(4, 11, 9)(6, 13, 10)(12, 16, 17)(15, 18, 19)$, and $t \sim t_1$. We factor the above progenitor by the relation $(xt)^3$, but first we expand $(xt)^3 = 1$ as follows:

Note: $x = (1, 2, 6, 12, 4)(3, 9, 15, 16, 10)(5, 13, 8, 14, 7)(11, 17, 19, 20, 18)$,

$x^2 = (1, 6, 4, 2, 12)(3, 15, 10, 9, 16)(5, 8, 7, 13, 14)(11, 19, 18, 17, 20)$, and

$x^3 = (1, 12, 2, 4, 6)(3, 16, 9, 10, 15)(5, 14, 13, 7, 8)(11, 20, 17, 18, 19)$.

$$(xt)^3 = 1$$

$$\implies xt_1 \cdot xt_1 \cdot xt_1 = 1.$$

$$\implies x^3 t_1 x^2 t_1 x t_1 = 1.$$

$$\implies (1, 12, 2, 4, 6)(3, 16, 9, 10, 15)(5, 14, 13, 7, 8)(11, 20, 17, 18, 19) t_6 t_2 t_1 = 1.$$

The expanded relation is:

$$(1, 12, 2, 4, 6)(3, 16, 9, 10, 15)(5, 14, 13, 7, 8)(11, 20, 17, 18, 19) t_6 t_2 t_1 = 1.$$

Factoring gives us the finite homomorphic image:

$$G \cong L(2, 11) \cong \frac{2^{*20} : A_5}{(1, 12, 2, 4, 6,) (3, 16, 9, 10, 15) (5, 14, 13, 7, 8) (11, 20, 17, 18, 19) t_6 t_2 t_1 = 1}$$

We wish to find the index of N in G by the double coset enumeration algorithm of

$G \cong L(2, 11)$ over $N \cong A_5$ as follows:

The First Double Coset $NeN=[*]$

Let NeN be denoted by $[*]$. $NeN = \{ Nen | n \in N \} = Ne^n = N$. N is the only single coset contained in the double coset $[*]$, thus N is transitive on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$. Therefore, it has the single orbit $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$. We choose 1 to be the representative for this orbit, and form the coset Nt_1 such that $Nt_1 \in Nt_1N = [1]$. Thus, Nt_1N , is a new double coset, so 20 symmetric generators extend to this new double coset $Nt_1N = \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9, Nt_{10}, Nt_{11}, Nt_{12}, Nt_{13}, Nt_{14}, Nt_{15}, Nt_{16}, Nt_{17}, Nt_{18}, Nt_{19}, Nt_{20}\}$.

The Second Double Coset $Nt_1N=[1]$

We wish to find the number of single cosets in the double coset Nt_1N . Now, the point-stabilizer of Nt_1N , $N^1 = \langle (2, 13, 18)(3, 5, 20)(4, 11, 14)(6, 12, 19)(7, 15, 17)(8, 16, 10) \rangle \cong \mathbb{Z}_3$.

Theorem: $t_1 = t_9$ We first state and prove 5 lemmas.

Lemma 1: $1\ 9 = 20\ 14$

Proof: $19 = 19$ (insert identity 15 15 and 33)

$$19 = 1\ 15\ 15\ 9\ 3\ 3$$

$$1\ 9 = 1\ 15\ \underline{15\ 9\ 3\ 3}$$

(conjugating relation by $(1, 15)(2, 9)(3, 6)(4, 16)(5, 11)(7, 17)(8, 20)(10, 12)(13, 18)(14, 19)$ gives $15 \ 9 \ 3 = (1, 6, 4, 2, 12)(3, 15, 10, 9, 16)(5, 8, 7, 13, 14)(11, 19, 18, 17, 20)$)

$= (1, 6, 4, 2, 12)(3, 15, 10, 9, 16)(5, 8, 7, 13, 14)(11, 19, 18, 17, 20) \underline{6} \ 10 \ 3$
 (conjugating relation by $(1, 6)(2, 14)(3, 8)(4, 19)(5, 11)(7, 10)(9, 16)(12, 18)(13, 17)(15, 20)$ gives $(1, 6, 18, 14, 19)(2, 17, 10, 3, 11)(4, 5, 15, 13, 12)(7, 20, 8, 9, 16)$
 $1 \ 14 = 6$

$= (1, 18, 10, 16, 11)(2, 4, 17, 8, 20)(3, 13, 19, 14, 15)(5, 9, 7, 12, 6) \ 1 \ \underline{14} \ \underline{10} \ 3$
 (conjugating relation by $(1, 6, 18, 14, 19)(2, 17, 10, 3, 11)(4, 5, 15, 13, 12)(7, 20, 8, 9, 16)$ gives $14 \ 10 = (1, 13, 16, 8, 4)(2, 12, 11, 7, 20)(3, 9, 17, 6, 19)(5, 18, 14, 15, 10) \ 18$)

$= (1, 14, 10, 8, 2)(3, 16, 7, 11, 13)(4, 6, 18, 5, 17)(9, 20, 12, 19, 15) \ 13 \ 18 \ \underline{3}$
 (conjugating relation by $(1, 13, 12, 19, 11)(2, 14, 7, 6, 3)(4, 15, 20, 18, 16)(5, 9, 17, 10, 8)$ gives $3 = (1, 11, 16, 10, 18)(2, 20, 8, 17, 4)(3, 15, 14, 19, 13)(5, 6, 12, 7, 9) \ 13 \ 14$)

$= (1, 19, 14, 18, 6)(2, 11, 3, 10, 17)(4, 12, 13, 15, 5)(7, 16, 9, 8, 20) \ 3 \ \underline{1} \ \underline{13} \ 14$
 (conjugating relation by $(2, 13, 18)(3, 5, 20)(4, 11, 14)(6, 12, 19)(7, 15, 17)(8, 16, 10)$ gives $1 \ 13 = (1, 19, 13, 11, 12)(2, 6, 14, 3, 7)(4, 18, 15, 16, 20)(5, 10, 9, 8, 17) \ 12$)

$= (1, 13, 16, 8, 4)(2, 12, 11, 7, 20)(3, 9, 17, 6, 19)(5, 18, 14, 15, 10) \ \underline{7} \ \underline{12} \ 14$ (conjugating relation by $(1, 20)(2, 12)(3, 5)(4, 11)(6, 7)(8, 19)(9, 14)(10, 15)(13, 17)(16, 18)$ gives $20 = (1, 13, 16, 8, 4)(2, 12, 11, 7, 20)(3, 9, 17, 6, 19)(5, 18, 14, 15, 10) \ 7 \ 12$)

$$= 20 \ 14. \quad \square$$

Lemma 2: $1 \ 9 = 7 \ 18$

Proof:

$$19 = 19 \text{ (insert identity } 15 \ 15 \text{ and } 3 \ 3 \text{)}$$

$$1 \ 9 = 1 \ 15 \ \underline{15 \ 9 \ 3 \ 3} \text{ (see lemma 1 for conjugation)}$$

$$= (1, 6, 4, 2, 12)(3, 15, 10, 9, 16)(5, 8, 7, 13, 14)(11, 19, 18, 17, 20) \ \underline{6 \ 10 \ 3}$$

(realtion gives $6 = (1,6,4,2,12)(3,15,10,9,16)(5,8,7,13,14)(11,19,18,17,20) \ 1 \ 2$)

$$= (1, 4, 12, 6, 2)(3, 10, 16, 15, 9)(5, 7, 14, 8, 13)(11, 18, 20, 19, 17) \ 1 \ 2 \ \underline{10 \ 3}$$

(conjugate relation by $(1, 9)(2, 3)(4, 15)(5, 18)(6, 10)(7, 11)(8, 19)(12, 16)(13, 20)(14, 17)$ gives $10 \ 3 = (1,12,2,4,6)(3,16,9,10,15)(5,14,13,7,8)(11,20,17,18,19) \ 9$)

$$= (1, 6, 4, 2, 12)(3, 15, 10, 9, 16)(5, 8, 7, 13, 14)(11, 19, 18, 17, 20) \ 12 \ \underline{4 \ 9}$$

(conjugate relation by $(1, 4, 18)(2, 5, 19)(3, 7, 9)(8, 15, 11)(10, 13, 20)(12, 17, 14)$ to obtain $4 = (1, 2, 8, 10, 14)(3, 13, 11, 7, 16)(4, 17, 5, 18, 6)(9, 15, 19, 12, 20)6 \ 5$)

$$= (1, 4, 8, 16, 13)(2, 20, 7, 11, 12)(3, 19, 6, 17, 9)(5, 10, 15, 14, 18) \ 20 \ 6 \ \underline{5 \ 9}$$

(conjugate relation by $(1, 17, 15)(2, 9, 13)(3, 12, 8)(4, 10, 19)(5, 14, 6)(11, 20, 16)$ to obtain $5 \ 9 = (1,19,13,11,12)(2,6,14,3,7)(4,18,15,16,20)(5,10,9,8,17) \ 17$)

$= (1, 18, 10, 16, 11)(2, 4, 17, 8, 20)(3, 13, 19, 14, 15)(5, 9, 7, 12, 6) \ 4 \ 14 \ \underline{17}$
 (conjugate relation by $(1, 17)(2, 18)(3, 16)(4, 6)(5, 12)(7, 15)(8, 19)(9, 13)(10, 11)(14, 20)$ to obtain $17 = (1, 2, 8, 10, 14)(3, 13, 11, 7, 16)(4, 17, 5, 18, 6)(9, 15, 19, 12, 20) \ 4 \ 18$)

$= (1, 6, 18, 14, 19)(2, 17, 10, 3, 11)(4, 5, 15, 13, 12)(7, 20, 8, 9, 16) \ 17 \ \underline{1 \ 4} \ 18$
 (conjugate relation by $(1, 12, 2, 4, 6)(3, 16, 9, 10, 15)(5, 14, 13, 7, 8)(11, 20, 17, 18, 19)$ to obtain $1 \ 4 = (1, 6, 4, 2, 12)(3, 15, 10, 9, 16)(5, 8, 7, 13, 14)(11, 19, 18, 17, 20) \ 12$)

$= (1, 4, 8, 16, 13)(2, 20, 7, 11, 12)(3, 19, 6, 17, 9)(5, 10, 15, 14, 18) \ \underline{20 \ 12} \ 18$
 (conjugate relation by $(1, 20)(2, 12)(3, 5)(4, 11)(6, 7)(8, 19)(9, 14)(10, 15)(13, 17)(16, 18)$ to obtain $20 \ 12 = (1, 13, 16, 8, 4)(2, 12, 11, 7, 20)(3, 9, 17, 6, 19)(5, 18, 14, 15, 10) \ 7$)

$= 7 \ 18. \square$

Lemma 3: $\underline{1 \ 9 = 9 \ 1}$

Proof: By Lemma 1 and Lemma 2, $1 \ 9 = 20 \ 14 = 7 \ 18$

Conjugating $20 \ 14 = 7 \ 18$ by $(1, 13, 14)(2, 16, 5)(3, 18, 12)(4, 7, 10)(6, 11, 15)(9, 17, 20)$, we have $9 \ 1 = 10 \ 12$ and we obtain $10 \ 12 = 7 \ 18$ when $20 \ 14 = 1 \ 9$ is conjugated by $(1, 7, 15)(2, 9, 18)(3, 19, 11)(4, 8, 5)(6, 20, 10)(12, 16, 14)$.

Now $9 \ 1 = 10 \ 12$ and $10 \ 12 = 7 \ 18$.

So $9 \ 1 = 7 \ 18$ but $7 \ 18 = 1 \ 9$ (Lemma 2). Therefore $1 \ 9 = 9 \ 1$. \square

Lemma 4: $1\ 9 = 5\ 11$

Proof: Conjugating $1\ 9 = 20\ 14$ (Lemma 1) by $(2, 18, 13)(3, 20, 5)(4, 14, 11)(6, 19, 12)(7, 17, 15)(8, 10, 16)$, we have $1\ 9 = 5\ 11$. \square

Lemma 5: $5\ 11 = 9\ 1$

Proof: $1\ 9 = 5\ 11$ (Lemma 4) and $1\ 9 = 9\ 1$ (Lemma 3). \square

Proof: We now prove $t_1 = t_9$

Conjugate Relation by $(1, 17, 15)(2, 9, 13)(3, 12, 8)(4, 10, 19)(5, 14, 6)(11, 20, 16)$ to obtain $(1, 12, 11, 13, 19)(2, 7, 3, 14, 6)(4, 20, 16, 15, 18)(5, 17, 8, 9, 10) = 17\ 9\ 5$

conjugating relation by $(1, 11, 2)(3, 8, 18)(4, 19, 7)(5, 15, 9)(6, 13, 20)(14, 16, 17)$ and obtain $(1, 19, 13, 11, 12)(2, 6, 14, 3, 7)(4, 18, 15, 16, 20)(5, 10, 9, 8, 17) = 11\ 1\ 13$

Thus $(1, 12, 11, 13, 19)(2, 7, 3, 14, 6)(4, 20, 16, 15, 18)(5, 17, 8, 9, 10) = 17\ 9\ 5$ and $(1, 19, 13, 11, 12)(2, 6, 14, 3, 7)(4, 18, 15, 16, 20)(5, 10, 9, 8, 17) = 11\ 1\ 13$.

Then, $(1, 12, 11, 13, 19)(2, 7, 3, 14, 6)(4, 20, 16, 15, 18)(5, 17, 8, 9, 10)(1, 19, 13, 11, 12)(2, 6, 14, 3, 7)(4, 18, 15, 16, 20)(5, 10, 9, 8, 17) = 17\ 9\ 5\ 11\ 1\ 13$

implies $17\ 9\ 5\ 11\ 1\ 13$

$\text{Id} = 17\ 9\ 9\ 1\ 1\ 13$ (Lemma 5)

implies $\text{Id} = 17\ 13$

Now conjugate $17\ 13 = \text{Id}$ by

$$(1, 17)(2, 18)(3, 16)(4, 6)(5, 12)(7, 15)(8, 19)(9, 13)(10, 11)(14, 20).$$

So $1\ 9 = \text{Id}$. This gives $1 = 9$.

Hence $t_1 = t_9$. \square

From the above theorem, we have $Nt_1 = Nt_9$.

Thus, $N(t_1)^{(1,9)(2,20)(3,18)(4,7)(5,13)(6,16)(8,12)(10,19)(11,17)(14,15)} = Nt_9 = Nt_1$. Thus, $(1, 9)(2, 20)(3, 18)(4, 7)(5, 13)(6, 16)(8, 12)(10, 19)(11, 17)(14, 15) \in N^{(1)}$. So, $N^{(1)} = \langle (2, 13, 18)(3, 5, 20)(4, 11, 14)(6, 12, 19)(7, 15, 17)(8, 16, 10), (1, 9)(2, 20)(3, 18)(4, 7)(5, 13)(6, 16)(8, 12)(10, 19)(11, 17)(14, 15) \rangle \cong S_3$. Thus, the number of right cosets in $Nt_1N = \frac{|60|}{|6|} = 10$.

The orbits of $N^{(1)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$ are $\{1, 9\}, \{2, 18, 13, 20, 5, 3\}, \{4, 14, 11, 7, 17, 15\}$, and $\{6, 19, 12, 16, 8, 10\}$. We choose 1, 2, 4, and 6 to be representatives from each orbit and form the cosets $Nt_1t_1, Nt_1t_2, Nt_1t_4$, and Nt_1t_6 . We determine which double coset they belong to.

$Nt_1t_1 \in N$:

$$Nt_1t_1 = Nt_1^2 = N.$$

Thus, 2 symmetric generators go back to $[*]$.

$Nt_1t_2 \in Nt_1N$:

relation $(1, 12, 2, 4, 6)(3, 16, 9, 10, 15)(5, 14, 13, 7, 8)(11, 20, 17, 18, 19)t_6t_2t_1 = 1$.

can be re-written as $t_1t_2 = (1, 12, 2, 4, 6)(3, 16, 9, 10, 15)(5, 14, 13, 7, 8)(11, 20, 17, 18, 19)t_6$

$\implies Nt_1t_2 = Nt_6$

$Nt_6 = N(t_1)^{(1,6)(2,14)(3,8)(4,19)(5,11)(7,10)(9,16)(12,18)(13,17)(15,20)}$

$\implies Nt_1t_2 \in Nt_1N$

Thus, 6 symmetric generators go back to [1].

$Nt_1t_4 \in Nt_1N$:

conjugate relation by $(1,12,2,4,6)(3,16,9,10,15)(5,14,13,7,8)(11,20,17,18,19)$

to obtain $(1,12,2,4,6)(3,16,9,10,15)(5,14,13,7,8)(11,20,17,18,19)t_1t_4 = t_{12}$

$\implies Nt_1t_4 = Nt_{12} = N(t_1)^{(1,12)(2,9)(3,17)(4,13)(5,16)(6,11)(7,18)(8,15)(9,10)(14,20)}$

$\implies Nt_1t_4 \in Nt_1N$

Thus, 6 symmetric generators go back to [1].

$Nt_1t_6 \in Nt_1N$:

$1\ 6 = 1\ 2\ \underline{2\ 6}$

conjugate relation by $(1, 2, 6, 12, 4)(3, 9, 15, 16, 10)(5, 13, 8, 14, 7)(11, 17, 19, 20, 18)$ to obtain $2\ 6 = (1, 12, 2, 4, 6)(3, 16, 9, 10, 15)(5, 14, 13, 7, 8)(11, 20, 17, 18, 19)\ 12$

$= (1, 12, 2, 4, 6)(3, 16, 9, 10, 15)(5, 14, 13, 7, 8)(11, 20, 17, 18, 19)\ \underline{12\ 4\ 12}$

conjugate relation by $(1, 12, 2, 4, 6)(3, 16, 9, 10, 15)(5, 14, 13, 7, 8)(11, 20, 17,$

18, 19) to obtain $12\ 4 = (1, 12, 2, 4, 6)(3, 16, 9, 10, 15)(5, 14, 13, 7, 8)(11, 20, 17, 18, 19)\ 1$

$= (1, 4, 12, 6, 2)(3, 10, 16, 15, 9)(5, 7, 14, 8, 13)(11, 18, 20, 19, 17)\ 1\ \underline{12}$
 conjugate relation by $(2, 13, 18)(3, 5, 20)(4, 11, 14)(6, 12, 19)(7, 15, 17)(8, 16, 10)$
 to obtain $12 = (1, 12, 11, 13, 19)(2, 7, 3, 14, 6)(4, 20, 16, 15, 18)(5, 17, 8, 9, 10)\ 1\ 13$

$= (1, 20)(2, 12)(3, 5)(4, 11)(6, 7)(8, 19)(9, 14)(10, 15)(13, 17)(16, 18)\ 12\ 1\ \underline{13}$
 conjugate relation by $(1, 13, 14)(2, 16, 5)(3, 18, 12)(4, 7, 10)(6, 11, 15)(9, 17, 20)$
 to obtain

$13 = (1, 14, 10, 8, 2)(3, 16, 7, 11, 13)(4, 6, 18, 5, 17)(9, 20, 12, 19, 15)\ 11\ 16$

$= (1, 12)(2, 19)(3, 17)(4, 13)(5, 16)(6, 11)(7, 18)(8, 15)(9, 10)(14, 20)\ 19\ 14\ \underline{11}\ \underline{16}$
 (1=9 implies 11=5 and 6=16)

$= (1, 12)(2, 19)(3, 17)(4, 13)(5, 16)(6, 11)(7, 18)(8, 15)(9, 10)(14, 20)\ 19\ 14\ \underline{5}\ \underline{6}$
 conjugate relation by $(1, 5, 20)(2, 6, 17)(7, 19, 10)(8, 12, 18)(9, 11, 14)(13, 15, 16)$
 to obtain

$5\ 6 = (1, 2, 8, 10, 14)(3, 13, 11, 7, 16)(4, 17, 5, 18, 6)(9, 15, 19, 12, 20)\ 17$

$= (1, 20)(2, 12)(3, 5)(4, 11)(6, 7)(8, 19)(9, 14)(10, 15)(13, 17)(16, 18)\ \underline{1}\ \underline{4}\ 17$
 conjugate relation by $(1, 12, 2, 4, 6)(3, 16, 9, 10, 15)(5, 14, 13, 7, 8)(11, 20, 17, 18, 19)$ to obtain $1\ 4 = (1, 6, 4, 2, 12)(3, 15, 10, 9, 16)(5, 8, 7, 13, 14)(11, 19, 18, 17, 20)\ 12$

$$= (1, 11, 2)(3, 8, 18)(4, 19, 7)(5, 15, 9)(6, 13, 20)(14, 16, 17) \underline{12} \ 17$$

(1=9 implies 12=10)

$$= (1, 11, 2)(3, 8, 18)(4, 19, 7)(5, 15, 9)(6, 13, 20)(14, 16, 17) \underline{10} \underline{17}$$

conjugate relation by $(1, 10, 11, 18, 16)(2, 17, 20, 4, 8)(3, 19, 15, 13, 14)(5, 7, 6, 9, 12)$ to obtain $10 \ 17 = (1, 12, 11, 13, 19)(2, 7, 3, 14, 6)(4, 20, 16, 15, 18)(5, 17, 8, 9, 10) \ 9$

$$= (1, 13, 16, 8, 4)(2, 12, 11, 7, 20)(3, 9, 17, 6, 19)(5, 18, 14, 15, 10) \ 9$$

$$= 1$$

Thus, 6 symmetric generators go back to $[1]$.

We, have completed the double coset enumeration of G over N .

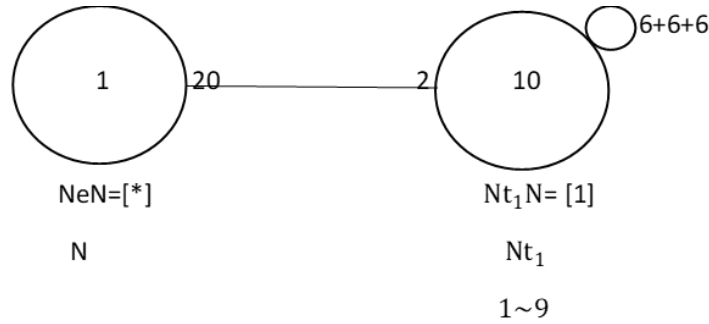


Figure 6.2: Cayley Graph of $L(2,11)$ over A_5

Chapter 7

Double Coset $U(3,5):2$

7.1 DCE $U(3,5):2$ over $PGL(2,9)$

A symmetric presentation of the progenitor $2^{*12}:PGL(2,9)$ is $\langle x, y, t \mid x^3, (xy^{-1})^2, y^{10}, (x^{-1}y^{-2})^4, y^3 x^{-1} y^{-4} x y^3 x y^{-4} x^{-1} y^2, t^2, (t,x), (t, y^2 x y^{-2}) \rangle$. $N = \langle x, y \rangle \cong PGL(2,9)$ of order 720, where $x \sim (2, 4, 10)(3, 5, 7)(6, 12, 8)$, $y \sim (1, 10, 9, 2, 7, 12, 11, 8, 5, 4)(3, 6)$, and $t \sim t_1$. We prove that the above progenitor factored by the relations $tt^{y^4x} = x y x y^2 x y^3 x y^4 x y^2 t t^{y^8}$, $(y^3 t^{(y^{-2}x^{-1})})^5$, and $(y^3 t)^8$ gives $U(3,5):2$.

We expand $(y^3 t_3)^5 = 1$.

$$\implies (y^3 t_3)^5 = 1$$

$$\implies (y^3 t_3)(y^3 t_3)(y^3 t_3)(y^3 t_3)(y^3 t_3) = 1$$

$$\implies (y^3)^5 t_3 (y^3)^4 t_3 (y^3)^3 t_3 (y^3)^2 t_3 y^3 t_3 = 1.$$

Thus, $(y^3)^5 t_3 t_6 t_3 t_6 t_3 = (1,12)(2,5)(3,6)(4,7)(8,9)(10,11)t_3 t_6 t_3 t_6 t_3 = 1$.

Similarly, the other two relations are expanded and we have

$$t_1 t_1^{y^4 x} = xyxy^2 xy^3 xy^4 xy^2 t_1 t_1^{y^8}$$

$$\implies t_1 t_3 = (1, 3, 5)(2, 12, 4)(6, 8, 10)t_1 t_5 \text{ and}$$

$$(y^3 t_1)^8$$

$$\text{Let } \pi = y^3$$

$$\implies (\pi t_1)^8 = \pi^8 t_1^{\pi^7} t_1^{\pi^6} t_1^{\pi^5} t_1^{\pi^4} t_1^{\pi^3} t_1^{\pi^2} t_1^{\pi} t_1.$$

$$\implies (1, 7, 5, 9, 11)(2, 8, 10, 12, 4)t_{10}t_5t_{12}t_9t_4t_{11}t_2t_1 = 1.$$

Therefore, the expanded relations are:

1. $(1, 3, 5)(2, 12, 4)(6, 8, 10)t_1 t_5 t_3 t_1 = 1$
2. $(1, 12)(2, 5)(3, 6)(4, 7)(8, 9)(10, 11)t_3 t_6 t_3 t_6 t_3 = 1$
3. $(1, 7, 5, 9, 11)(2, 8, 10, 12, 4)t_{10}t_5t_{12}t_9t_4t_{11}t_2t_1 = 1$

These relations give us the finite homomorphic image:

$$G \cong U(3, 5) : 2 \cong \frac{2^{*12} : PGL(2, 9)}{(1, 3, 5)(2, 12, 4)(6, 8, 10)t_1 t_5 t_3 t_1 = 1, \\ (1, 12)(2, 5)(3, 6)(4, 7)(8, 9)(10, 11)t_3 t_6 t_3 t_6 t_3 = 1, \\ (1, 7, 5, 9, 11)(2, 8, 10, 12, 4)t_{10}t_5t_{12}t_9t_4t_{11}t_2t_1 = 1}$$

We wish to find the index of N in G by the double coset enumeration algorithm of $G \cong U(3, 5) : 2$ over $N \cong PGL(2, 9)$ as follows:

The First Double Coset $NeN = [*]$

Let NeN be denoted by $[*]$. $NeN = \{ Nen \mid n \in N \} = Ne^n = N$. N is the only single coset contained in the double coset $[*]$, thus N is transitive on $\{1,$

$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and has the single orbit $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. We choose 1 to be the representative from this orbit and form the coset Nt_1 such that $Nt_1 \in Nt_1N = [1]$. Nt_1N , is a new double coset, so 12 symmetric generators extend to the double coset $Nt_1N = \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9, Nt_{10}, Nt_{11}, Nt_{12}\}$.

The Second Double Coset $Nt_1N = [1]$

We wish to find the number of single cosets in the double coset Nt_1N . We find the point-stabilizer of 1, denoted by N^1 . Now, $N^{(1)} \geq N^1 = \langle (2, 4, 10)(3, 5, 7)(6, 12, 8), (2, 12, 6)(3, 11, 9)(4, 10, 8) \rangle \cong A_5$ is of order 60. Therefore, the number of single cosets in Nt_1N is given by $\frac{|N|}{|N^{(1)}|} = \frac{720}{60} = 12$.

The orbits of $N^{(1)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are $\{1\}, \{3, 5, 11, 7, 9\}$, and $\{2, 4, 12, 10, 8, 6\}$. We choose 1, 2, and 3 to be representatives from each orbit and form the cosets $Nt_1t_1, Nt_1t_3, Nt_1t_2$. Now, $Nt_1t_1 = Nt_1^2 = N$, so 1 symmetric generator goes back to $[*]$. We note that $Nt_1t_2 \in Nt_1t_2N$ and $Nt_1t_3 \in Nt_1t_3N$ are new double cosets. Thus 6 symmetric generators extend to Nt_1t_2N and 5 symmetric generators extend to Nt_1t_3N .

The Third Double Coset $Nt_1t_2N = [12]$

We wish to find the number of single cosets in the double coset Nt_1t_2N . We find the point-stabilizer of the double coset Nt_1t_2N , denoted by N^{12} . Now the coset stabilizer $N^{(12)} \geq N^{12} = \langle (3, 7, 11, 9, 5)(4, 6, 12, 10, 8), (4, 10)(5, 7)(6,$

$12)(9, 11)>$ is of order 10. So, the number of single cosets in Nt_1t_2N is given by $\frac{|N|}{|N^{(12)}|} = \frac{720}{10} = 72$. The orbits of $N^{(12)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are $\{1\}$, $\{2\}$, $\{3, 7, 11, 9, 5\}$, and $\{4, 6, 10, 12, 8\}$. We Choose 1, 2, 3 and 4 to be representatives from each orbit and form the cosets $Nt_1t_2t_1$, $Nt_1t_2t_2$, $Nt_1t_2t_3$, and $Nt_1t_2t_4$.

Now, $Nt_1t_2t_2 = Nt_1t_2^2 = Nt_1 \in [1]$,

So, 1 symmetric generator goes back to $[1]$.

$Nt_1t_2t_1 \in Nt_1t_2N$:

relation 2 can be re-written as $(1, 12)(2, 5)(3, 6)(4, 7)(8, 9)(10, 11)t_3t_6t_3 = t_3t_6$

conjugate relation 2 by $(1, 3)(2, 6)(5, 11)(8, 12)$ gives

$$t_1t_2t_1 = (1, 2)(3, 8)(4, 7)(5, 10)(6, 11)(9, 12)t_1t_2$$

$$\implies Nt_1t_2t_1 = N(1, 2)(3, 8)(4, 7)(5, 10)(6, 11)(9, 12)t_1t_2$$

$$\implies Nt_1t_2t_1 = Nt_1t_2$$

Thus, $Nt_1t_2t_1 \in [12]$ and 1 symmetric generator goes back to $[12]$.

$$Nt_1t_2t_3 \in Nt_1t_2t_3N = [123]$$

Thus, 5 symmetric generators extend to $Nt_1t_2t_3N = [123]$

$$Nt_1t_2t_4 \in Nt_1t_2t_4N = [124]$$

Thus, 5 symmetric generators extend to $Nt_1t_2t_4N = [124]$.

The Fourth Double Coset $Nt_1t_3N = [13]$

We wish to find the single cosets of the double coset Nt_1t_3N . Now $N^{13} = <(2, 6, 10)(4, 8, 12)(5, 11, 9), (4, 10)(5, 7)(6, 12)(9, 11), (2, 8)(5, 11)(6, 12)(7, 9), > \cong 2^2 : 3$.

Lemma $t_1t_3 = t_1t_7$:

Proof:

By relation 1, $(1, 3, 5)(2, 12, 4)(6, 8, 10)t_1t_5 = t_1t_3$

we have $t_1t_3 = t_1t_5$.

Now, conjugate relation 1 by $(2, 4, 8)(3, 7, 9)(6, 12, 10)$ gives

$$(1, 7, 5)(2, 6, 12)(4, 10, 8)t_1t_5 = t_1t_7$$

but, $t_1t_5 = t_1t_3$ (see previous)

Therefore, $Nt_1t_3 = Nt_1t_7$. \square

Now, $N(t_1t_3)^{(2,8,6)(3,7,11)(4,10,12)} = Nt_1t_7 = Nt_1t_3$ implies $(2, 8, 6)(3, 7, 11)(4, 10, 12) \in N^{(13)}$. Thus, $N^{(13)} \geq < N^{13}, (2, 8, 6)(3, 7, 11)(4, 10, 12) > = < (2, 8, 6)(3, 7, 11)(4, 10, 12), (4, 10)(5, 7)(6, 12)(9, 11), (2, 8)(5, 11)(6, 12)(7, 9), (2, 8, 6)(3, 7, 11)(4, 10, 12) > \cong A_5$.

Since $[13]^{N^{(13)}}$ gives $\{[13], [17], [19], [1, 11]\}$, we have $13 \sim 17 \sim 19 \sim 1$. Thus, the number of single cosets in Nt_1t_3N is $\frac{720}{60} = 12$. The orbits of $N^{(13)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are $\{1\}$, $\{3, 5, 7, 11, 9\}$, and $\{2, 4, 6, 10, 12, 8\}$. We choose 1, 2, and 3 to be representatives from each orbit and form the cosets $Nt_1t_3t_1$, $Nt_1t_3t_3$, and $Nt_1t_3t_2$. We determine their double cosets.

$Nt_1t_3t_2 \in [124]$:

$$\begin{aligned} \text{We have } Nt_1t_3t_2 &= (1, 9, 3, 5)(2, 8, 10, 4)(6, 12)(7, 11) t_5t_2t_{10} \\ &= (1, 9, 3, 5)(2, 8, 10, 4)(6, 12)(7, 11)(Nt_1t_2t_4)^{(1,5,7)(3,9,11)(4,10,8)}. \end{aligned}$$

So, 6 symmetric generators extend to $[124]$.

$Nt_1t_3t_3$ in $[1]$:

$$Nt_1t_3t_3 = Nt_1t_3^2 = Nt_1 \in [1].$$

So, 5 symmetric generators go back to $[1]$.

$$Nt_1t_3t_1 \in Nt_1t_3t_1N = [131]$$

Thus, 1 symmetric generator extend to $Nt_1t_3t_1N = [131]$.

The Fifth Double Coset $Nt_1t_2t_3N = [123]$

We wish to find the number of single cosets in the double coset $Nt_1t_2t_3N$.

The point stabilizer $N^{123} = \langle (4, 10)(5, 7)(6, 12)(9, 11) \rangle$.

Lemma: $t_1t_2t_3 = t_2t_1t_8$

Proof:

$$Nt_2t_1t_8 = N(1, 3)(2, 8)(4, 12, 10, 6)(5, 9, 7, 11)t_1t_2t_3$$

Thus, $Nt_1t_2t_3 = Nt_2t_1t_8$. \square

Now, $N(t_1t_2t_3)^{(1,2)(3,8)(4,5)(6,9)(7,10)(11,12)} = Nt_2t_1t_8 = Nt_1t_2t_3$ implies $(1, 2)(3, 8)(4, 5)(6, 9)(7, 10)(11, 12) \in N^{(123)} = \geq \{N^{123}, (1, 2)(3, 8)(4, 5)(6, 9)(7,$

$10)(11, 12)\} \cong 2^2$. Since $[123]^{N^{(123)}}$ gives $\{[123], [218]\}$, we have $123 \sim 218$. Thus, the number of single cosets in $Nt_1t_2t_3N$ is given by $\frac{|N|}{|N^{(123)}|} = \frac{720}{4} = 180$.

The orbits of $N^{(123)}$ on $\{1,2,3,4,5,6,7,8,9,10,11,12\}$ are $\{1,2\}$, $\{3,8\}$, $\{6,9,11,12\}$, $\{4,5,7,10\}$. We choose representatives 1, 3, 4, and 6 from each orbit and form the cosets $Nt_1t_2t_3t_1$, $Nt_1t_2t_3t_3$, $Nt_1t_2t_3t_4$, and $Nt_1t_2t_3t_6$.

$Nt_1t_2t_3t_3 \in [12]$:

$$Nt_1t_2t_3t_3 = Nt_1t_2t_3^2 = Nt_1t_2 \in [12].$$

So, 2 symmetric generators go back to $[12]$.

$Nt_1t_2t_3t_1 \in [124]$:

$$\begin{aligned} Nt_1t_2t_3t_1 &= (1, 10, 9, 6)(2, 7, 8, 11, 12, 5, 4, 3)t_7t_8t_{10} \\ &= (1, 10, 9, 6)(2, 7, 8, 11, 12, 5, 4, 3)N(t_1t_2t_4)^{(1,7,3,5)(2,8)(4,10,6,12)(9,11)} \in [124]. \end{aligned}$$

Thus, $Nt_1t_2t_3t_1 \in [124]$ and 2 symmetric generators go back to $[124]$,

$Nt_1t_2t_3t_4 \in [123]$:

$$\begin{aligned} Nt_1t_2t_3t_4 &= (1,12,11,8,9,4,3,2)(10,7,6,5) t_{11}t_8t_9 \\ &= (1,12,11,8,9,4,3,2)(10,7,6,5)N(t_1t_2t_3)^{(1,11)(2,8)(3,9)(6,10)} \in [123]. \end{aligned}$$

So, 4 symmetric generators go back to $[123]$.

$Nt_1t_2t_3t_6 \in [123]$:

$$\begin{aligned} Nt_1t_2t_3t_6 &= (1, 4, 5, 8, 7, 6, 3, 2)(9, 12, 11, 10) t_5t_8t_7 \\ &= (1, 4, 5, 8, 7, 6, 3, 2)(9, 12, 11, 10) N(t_1t_2t_3)^{(1,5)(2,8)(3,7)(10,12)} \in [123]. \end{aligned}$$

So, 4 symmetric generators go back to [123].

Since no new double cosets are formed, this double coset is closed under coset multiplication.

The Sixth Double Coset $Nt_1t_2t_4N=[124]$

We wish to find the number of single cosets in the double coset $Nt_1t_2t_4N$.

The point stabilizer is $N^{124} = \langle (3, 11)(5, 9)(6, 8)(10, 12) \rangle$.

Lemma: $t_1t_2t_4 = t_5t_2t_8$

Proof:

$$\begin{aligned} Nt_5t_2t_8 &= (1,9,7)(10,2,8)(5,11,3)t_{11}t_2t_{10} \\ &= (1,9,7)(10,2,8)(5,11,3)(Nt_1t_2t_4)^{(1,11)(3,5)(4,10)(8,12)}. \end{aligned}$$

Thus, $Nt_1t_2t_4 = Nt_5t_2t_8$. \square

Now, $N(t_1t_2t_4)^{(1,5)(3,9)(4,8)(6,12)} = Nt_5t_2t_8$. Then $(1, 5)(3, 9)(4, 8)(6, 12)$ belongs to $N^{(124)} \geq \langle N^{124}, (1, 5, 3, 11, 9)(4, 8, 12, 10, 6) \rangle = \langle (3, 11)(5, 9)(6, 8)(10, 12), (1, 5, 3, 11, 9)(4, 8, 12, 10, 6) \rangle \cong D_5$. Since $[124]^{N^{(124)}}$ gives $\{[124], [926], [3,2,12], [528], [11,2,10]\}$, we have $124 \sim 3\ 2\ 12 \sim 528 \sim 11\ 2\ 10$. Thus, the number of single cosets in $Nt_1t_2t_4N$ is given by $\frac{|N|}{|N^{(124)}|} = \frac{720}{10} = 72$.

The orbits of $N^{(124)}$ on $\{1,2,3,4,5,6,7,8,9,10,11,12\}$ are $\{2\}, \{7\}, \{1, 9, 11, 3, 5\}$, and $\{4, 6, 12, 10, 8\}$. We choose 1, 2, 4, and 7 to be representatives from each orbit and form the cosets $Nt_1t_2t_4t_1$, $Nt_1t_2t_4t_2$, $Nt_1t_2t_4t_4$, and $Nt_1t_2t_4t_7$. We determine which double coset each belongs to.

$Nt_1t_2t_4t_1 \in [123]$:

$$\begin{aligned} Nt_1t_2t_4t_1 &= (1,2)(10,9)(4,7)(5,12)(3,6)(8,11)t_4t_1t_2 \\ &= (1,2)(10,9)(4,7)(5,12)(3,6)(8,11)N(t_1t_2t_3)^{(1,4,5,8,7,6,3,2)(9,12,11,10)} \in [1, 2, 3]. \end{aligned}$$

So, 5 symmetric generators go back to $[123]$

$Nt_1t_2t_4t_2 \in [13]$:

$$\begin{aligned} Nt_1t_2t_4t_2 &= (1,11,3,7)(10,2,8,4)(9,5)(12,6)t_7t_{11} \\ &= (1,11,3,7)(10,2,8,4)(9,5)(12,6)N(t_1t_3)^{(1,7)(3,11)(6,12)(8,10)} \in [13]. \end{aligned}$$

So, 1 symmetric generator goes back to $[13]$.

$Nt_1t_2t_4t_4 \in [12]$:

$$Nt_1t_2t_4t_4 = Nt_1t_2t_4^2 = Nt_1t_2 \in [12].$$

So, 5 symmetric generators go back to $[12]$.

$Nt_1t_2t_4t_7 \in [124]$:

$$\begin{aligned} Nt_1t_2t_4t_7 &= (1,12,7,4,3,10,5,6)(2,9,8,11)t_3t_2t_{12} \\ &= (1,12,7,4,3,10,5,6)(2,9,8,11)N(t_1t_2t_4)^{(1,3)(4,12)(6,10)(9,11)} \in [124]. \end{aligned}$$

So, 1 symmetric generator goes back to $[124]$.

The Seventh Double Coset $Nt_1t_3t_1N = [131]$

We wish to find the number of single cosets in the double coset $Nt_1t_3t_1N$.

We find the point stabilizer to be $N^{131} = \langle (2, 6, 10)(4, 8, 12)(5, 11, 9), (2, 4, 12)(6, 8, 10)(7, 11, 9), (4, 10)(5, 7)(6, 12)(9, 11) \rangle$.

Lemma $t_1 t_3 t_1 = t_1 t_7 t_1$

Proof:

re-write relation 1 as $(1, 3, 5)(2, 12, 4)(6, 8, 10)t_1 t_5 = t_1 t_3$

multiply on the right by t_1

$$\implies t_1 t_3 t_1 = (1, 3, 5)(2, 12, 4)(6, 8, 10)\underline{t_1 t_5} t_1$$

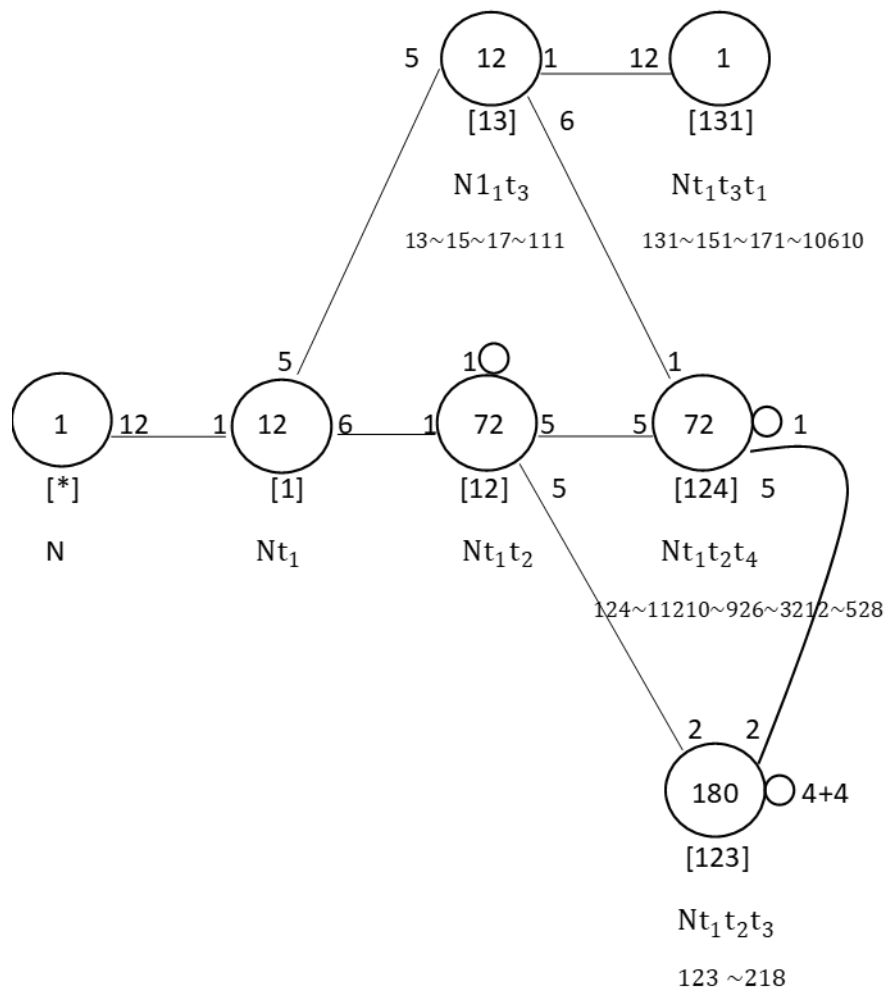
by previous proof we have $t_1 t_5 = t_1 t_7$

$$\implies t_1 t_3 t_1 = (1, 3, 5)(2, 12, 4)(6, 8, 10)t_1 t_7 t_1$$

$$\implies N t_1 t_3 t_1 = N t_1 t_7 t_1. \square$$

Now, $N(t_1 t_3 t_1)^{(2,6)(3,7)(4,12)(5,9)} = N t_1 t_7 t_1$ implies $(2, 6)(3, 7)(4, 12)(5, 9) \in N^{(131)} \geq \langle N^{131}, (2, 6)(3, 7)(4, 12)(5, 9) \rangle \cong \text{PGL}(2,9)$. Since $[131]^{N^{(131)}}$ gives $\{[131], [171], [151], [6,10,6]\}$, we have $131 \sim 171 \sim 151 \sim 10\ 6\ 10$. Thus, the number of single cosets in $N t_1 t_3 t_1 N$ is given by $\frac{|N|}{|N^{(131)}|} = \frac{720}{720} = 1$.

The orbit of $N^{(131)}$ on $\{1,2,3,4,5,6,7,8,9,10,11,12\}$ is $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. We choose 1 to be the representative from the orbit to form the coset $N t_1 t_3 t_1 t_1$. Now, $N t_1 t_3 t_1 t_1 = N t_1 t_3 t_1^2 = N t_1 t_3 \in [13]$ so 12 symmetric generators go back to [13]. Thus, we have completed the manual double coset enumeration of G over N .

Figure 7.1: Cayley Graph of $U(3,5):2$ over $PGL(2,9)$

7.2 DCE of $U(3:5):2$ over Maximal Subgroup $Aut(6)$ and $PGL(2,9)$

We begin by factoring the progenitor $2^{*12}: PGL(2,9)$ by the relations

$$\begin{aligned} (yt^{y^{-2}x^{-1}})^5 &\implies ((1, 10, 9, 2, 7, 12, 11, 8, 5, 4)(3, 6)t_3)^5, \\ (yt)^8 &\implies ((1, 10, 9, 2, 7, 12, 11, 8, 5, 4)(3, 6)t_1)^8, \\ (y^3t^{y^{-2}x^{-1}})^{10} &\implies ((1, 2, 11, 4, 9, 12, 5, 10, 7, 8)(3, 6)t_3)^{10}, \text{ and} \\ (y^3t)^8 &\implies ((1, 2, 11, 4, 9, 12, 5, 10, 7, 8)(3, 6)t_1)^8 \end{aligned}$$

to obtain $U(3,5):2$.

The expanded relations are:

1. $(1,12)(2,5)(3,6)(4,7)(8,9)(10,11)t_3t_6t_3t_6t_3 = 1.$
2. $(1,5,11,7,9)(2,10,4,8,12)t_8t_{11}t_{12}t_7t_2t_9t_{10}t_1 = 1.$
3. $(1,7,5,9,11)(2,8,10,12,4)t_{10}t_5t_{12}t_9t_4t_{11}t_2t_1 = 1.$
4. $t_6t_3t_6t_3t_6t_3t_6t_3t_6t_3 = 1.$

where $x \sim (2, 4, 10)(3, 5, 7)(6, 12, 8)$, $y \sim (1, 10, 9, 2, 7, 12, 11, 8, 5, 4)(3, 6)$, and $t \sim t_1$.

Let H be a subgroup generated by $N = PGL(2,9)$ and $ty^2ty^{-2}t$. Then $H = \langle N, ty^2ty^{-2}t \rangle$ is maximal in G and $|H| = 1,440$. We now proceed with the double coset enumeration of G over H and N .

The First Double Coset $HeN = [*]$

Let the double coset HeN be denoted by $[*]$. N is transitive on $\{1, 2,$

3, 4, 5, 6, 7, 8, 9, 10, 11, 12 }. Therefore, N has the single orbit $\{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 \}$. Therefore, the number of single cosets in $[*]$ is given by $\frac{|N|}{|N|} = \frac{720}{720} = 1$. We take 1 to be the representative for this orbit and form the coset Ht_1 . Then, $Ht_1 \in Ht_1N = \{Ht_1, Ht_2, Ht_3, Ht_4, Ht_5, Ht_6, Ht_7, Ht_8, Ht_9, Ht_{10}, Ht_{11}, Ht_{12}\} = [1]$. So, 12 symmetric generators extend to the double coset $Ht_1N = [1]$.

The Second Double Coset $Ht_1N = [1]$

We wish to find the number of single cosets contained in the double coset Ht_1N . Now, the point stabilizer $N^1 = \langle (2, 4, 10)(3, 5, 7)(6, 12, 8)(2, 12, 6)(3, 11, 9)(4, 10, 8) \rangle$. Thus, $N^{(1)} \geq N^1 = \langle (2, 4, 10)(3, 5, 7)(6, 12, 8)(2, 12, 6)(3, 11, 9)(4, 10, 8) \rangle \cong A_5$ of order 60. Therefore, the number of single cosets in Ht_1N is given by $\frac{|N|}{|N^{(1)}|} = \frac{720}{60} = 12$.

The orbits of $N^{(1)}$ on $\{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 \}$ are $\{ 1 \}$, $\{ 3, 5, 11, 7, 9 \}$, $\{ 2, 4, 12, 10, 8, 6 \}$. We take 1, 2, and 3 to be representatives from each orbit and form the cosets Ht_1t_1 , Ht_1t_2 , and Ht_1t_3 . We determine their double cosets.

$Ht_1t_1 \in [*]$:

$$Ht_1t_1 = Ht_1^2 = He = H \in [*].$$

Thus, 1 symmetric generator goes back to $[*]$.

$Ht_1t_3 \in [1]$:

$$\begin{aligned} Ht_1t_3 &= t_1t_5t_1(1, 5, 3)(10, 8, 6)(4, 12, 2)(t_1)^{Id} \\ &= (1, 5, 3)(10, 8, 6)(4, 12, 2)Ht_5t_3t_5t_1 \\ &= Ht_1 \in [1]. \end{aligned}$$

Thus, 5 symmetric generators go back to $[1]$.

$Ht_1t_2 \in [12]$:

$$Ht_1t_2 \in Ht_1t_2t_1N = [12].$$

Thus, 6 symmetric generators extend to $Ht_1t_2N = [12]$.

The Third Double Coset $Ht_1t_2N = [12]$

Here, $N^{(12)} \geq N^{12}$, Thus, $N^{(12)} = N^{12} = \langle (3, 7, 11, 9, 5)(4, 6, 12, 10, 8), (4, 10)(5, 7)(6, 12)(9, 11), (3, 7, 11, 9, 5)(4, 6, 12, 10, 8), (3, 11, 5, 7, 9)(4, 12, 8, 6, 10), (3, 9, 7, 5, 11)(4, 10, 6, 8, 12), (3, 5, 9, 11, 7)(4, 8, 10, 12, 6), (4, 10)(5, 7)(6, 12)(9, 11), (3, 7)(4, 8)(5, 11)(6, 10), (3, 11)(5, 9)(6, 8)(10, 12), (3, 9)(4, 6)(7, 11)(8, 12), (3, 5)(4, 12)(7, 9)(8, 10) \rangle$ is of order 10. The number of single cosets in Ht_1t_2N is given by $\frac{|N|}{|N^{(12)}|} = \frac{720}{10} = 72$.

The orbits of $N^{(12)}$ on $\{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 \}$ are $\{1\}$, $\{2\}$, $\{3, 5, 11, 7, 9\}$, $\{4, 12, 10, 8, 6\}$. We choose one representative from each orbit and form the cosets $Ht_1t_2t_1$, $Ht_1t_2t_2$, $Ht_1t_2t_3$, and $Ht_1t_2t_4$. We determine their double cosets.

$Ht_1t_2t_1 \in [12]$:

relation 1 can be re-written as $(1,12)(2,5)(3,6)(4,7)(8,9)(10,11)t_3t_6t_3 = t_3t_6$

conjugate relation 1 by $(1,3)(2,6)(5,11)(8,12)$ this gives us

$$t_1t_2t_1 = (1,2)(3,8)(4,7)(5,10)(6,11)(9,12)t_1t_2$$

$$\implies Ht_1t_2t_1 = H(1,2)(3,8)(4,7)(5,10)(6,11)(9,12)t_1t_2$$

$$\implies Ht_1t_2t_1 = Ht_1t_2 \in [12].$$

Thus, 1 symmetric generator goes back to $[12]$.

$Ht_1t_2t_2 \in [1]$:

$$Ht_1t_2t_2 = Ht_1t_2^2 = Ht_1 \in [1].$$

Thus, 1 symmetric generator goes back to $[1]$.

$$Ht_1t_2t_3 \in Ht_1t_2t_3N = [123].$$

Thus, 5 symmetric generators extend to $[123]$.

$Ht_1t_2t_4 \in [12]$:

$$Ht_1t_2t_4 = t_1t_5t_1(1,9,5,7)(10,4,6,2)(3,11)(8,12)(Ht_1t_2)^{(1,7,9)(4,6,12)(5,11,3)}$$

$$= (1,9,5,7)(10,4,6,2)(3,11)(8,12)t_9t_7t_9t_7t_2$$

$$= Ht_1t_2, \text{ since } (1,9,5,7)(10,4,6,2)(3,11)(8,12)Ht_9t_7t_9 \in H.$$

$$= Ht_1t_2 \in [12]$$

Thus, 5 symmetric generators goes back to $[12]$.

The Fourth Double Coset $Ht_1t_2t_3N = [123]$

We wish to find the number of single cosets in the double coset $Ht_1t_2t_3N$.

We find the point stabilizer of 1,2 and 3, $N^{123} = \langle (4, 10)(5, 7)(6, 12)(9, 11) \rangle \cong \mathbb{Z}_2$.

Lemma: $Ht_1t_2t_3 = Ht_2t_1t_8$

Proof:

$$Ht_2t_1t_8 = (1, 3)(2, 8)(4, 12, 10, 6)(5, 9, 7, 11)H(t_1t_2t_3)^{(4,10)(5,7)(6,12)(9,11)}$$

Thus, $Ht_2t_1t_8 = Ht_1t_2t_3$. \square

Thus, $H(t_1t_2t_3)^{(1,2)(3,8)(4,7)(5,10)(6,11)(9,12)} = Ht_2t_1t_8 = Ht_1t_2t_3$ implies $(1, 2)(3, 8)(4, 7)(5, 10)(6, 11)(9, 12)$ is in the coset stabilizer $N^{(123)}$. Now, $N^{(123)} \geq \langle N^{123}, (1, 2)(3, 8)(4, 7)(5, 10)(6, 11)(9, 12) \rangle \cong 2$.

Lemma: $Ht_1t_2t_3 = Ht_3t_8t_1$

Proof:

$$\begin{aligned} Ht_3t_8t_1 &= t_1t_5t_1(1, 12, 3, 8, 7, 6, 11, 10)(2, 9, 4, 5)(t_1t_2t_3)^{(1,2)(10,5)(4,7)(9,12)(3,8)(6,11)} \\ &= (1, 12, 3, 8, 7, 6, 11, 10)(2, 9, 4, 5)t_{12}t_2t_{12}t_2t_1t_8 \\ &= Ht_1t_2t_3. \text{ So, } Ht_1t_2t_3 = Ht_3t_8t_1. \square \end{aligned}$$

Thus, $H(t_1t_2t_3)^{(1,3)(2,8)(4,6,10,12)(5,11,7,9)} = Ht_3t_8t_1$ this implies $(1, 3)(2, 8)(4, 6, 10, 12)(5, 11, 7, 9)$ belongs to $N^{(123)}$. Now, $N^{(123)} \geq \langle N^{123}, (1, 2)(3, 8)(4, 7)(5, 10)(6, 11)(9, 12), (1, 3)(2, 8)(4, 6, 10, 12)(5, 11, 7, 9) \rangle$. Since $[123]^{N^{(123)}}$ gives $\{[123], [832], [281], [381]\}$, we have $123 \sim 218 \sim 381 \sim 832$. Thus, the number of cosets in $Ht_1t_2t_3N$ is given by $\frac{|N|}{|N^{(123)}|} = \frac{720}{8} = 90$.

The orbits of $N^{(123)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are $\{1, 2, 8, 3\}$ and $\{4, 10, 5, 7, 11, 9, 6, 12\}$. We choose one representative from each orbit and form the double cosets $Nt_1t_2t_3t_3$ and $Nt_1t_2t_3t_4$. We determine their double coset.

$Ht_1t_2t_3t_3 \in [12]$:

$$Ht_1t_2t_3t_3 = Ht_1t_2t_3^2 = Ht_1t_2 \in [12].$$

Thus, 4 symmetric generators go back to $[12]$.

$Ht_1t_2t_3t_4 \in [123]$:

$$\begin{aligned} Ht_1t_2t_3t_4 &= H(1, 5, 3)(10, 8, 6)(4, 12, 2)t_4t_3t_4t_8t_{11}t_2 \\ &= (1, 5, 3)(10, 8, 6)(4, 12, 2)(Ht_1t_2t_3)^{(1, 8, 9, 10, 7, 6, 3, 2, 11, 12)} \end{aligned}$$

Thus, $Ht_1t_2t_3t_4 = Ht_1t_2t_3 \in [123]$.

Thus, 8 symmetric generators go back to $[123]$ and have completed the double coset enumeration of G over H and N .

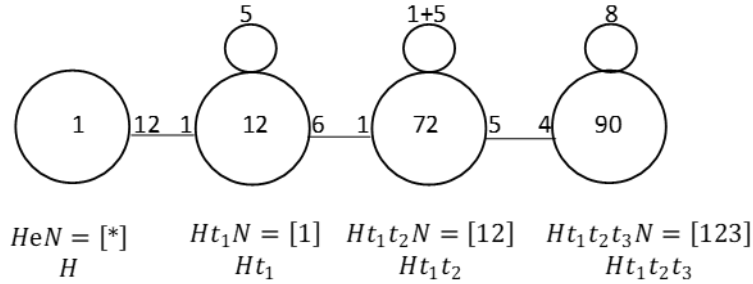


Figure 7.2: Cayley Graph DCE $U(3,5):2$ over Maximal Subgroup $Aut(6)$ and $PGL(2,9)$

Chapter 8

Monomial Representations

8.1 DCE PSL(2,11) over A_4

We are given a group $G \cong A_4 = \langle (1,2)(3,4), (1,2,3) \rangle$ and a subgroup $H = \langle (1,2)(3,4), (1,3)(2,4) \rangle$. If we induce the character $\chi^2 = [1 \ -1 \ 1 \ -1]$ of H up to the character $\chi^4 = [3 \ -1 \ 0 \ 0]$ of G , it gives us the monomial representation of G over \mathbb{Z}_3 . Inducing gives the following matrix representation:

$$A(x) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \text{ where } a_{11} = 2, a_{22} = 1, \text{ and } a_{33} = 2.$$

Since, $a_{ij} \iff t_i = t_j^a$, implies $t_1 \longrightarrow t_1^2$, $t_2 \longrightarrow t_2$, and $t_3 \longrightarrow t_3^2$.

$$A(y) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \text{ where } a_{12} = 1, a_{23} = 1, \text{ and } a_{31} = 1.$$

$t_1 \longrightarrow t_2$, $t_2 \longrightarrow t_3$, and $t_3 \longrightarrow t_1$.

The entries of each matrix are in \mathbb{Z}_3 , however we work in the cyclotomic field $\mathbb{Z}_3-1 = 2$. This gives a progenitor of the form $(Field\ of\ Entries)^{(*Dim\ of\ Matrix)} :_m N$. So far we have $\langle A(x), A(y) \rangle = 3^{*3} :_m G$. We wish to represent G as permutations, to do this we will look for the permutation representation of $A(x)$ and $A(y)$. The t_i 's are of order 3 and each t_i has 2 distinct powers. We label the t_i 's below.

Table 8.1: Labeling for $A(x)$

1	2	3	4	5	6
t_1	t_2	t_3	t_1^2	t_2^2	t_3^2
\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow
t_1^2	t_2	t_3^2	t_1	t_2^2	t_3
4	2	6	1	5	3

Table 8.2: Labeling for $A(y)$

1	2	3	4	5	6
t_1	t_2	t_3	t_1^2	t_2^2	t_3^2
\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow
t_2	t_3	t_1	t_2^2	t_3^2	t_1^2
2	3	1	5	6	4

These matrices yield permutations $A(x)=(1, 4)(3, 6)$ and $A(y)=(1, 2, 3)(4, 5, 6)$. So, G has a monomial representation $A(x), A(y)$ whose permutation representation is $G = \langle (1,4)(3,6), (1,2,3)(4,5,6) \rangle$. The normalizer of t is found by the powers of t_1 , we can see from the labeling that the powers of $\langle t \rangle = \{t_1, t_1^2, \text{Id}\}$, and the t_i 's are labeled as:

$$t_1 \longrightarrow 1, t_2 \longrightarrow 2, t_3 \longrightarrow 3, t_1^2 \longrightarrow 4, t_2^2 \longrightarrow 5, \text{ and } t_3^2 \longrightarrow 6.$$

Then $\{t_1, t_1^2\}^n = \{t_1, t_1^2\}$ this implies $\{1, 4\}^n = \{1, 4\}$. Now, we can have permutations that send this group to itself, so we can have permutations which send 1 to 1, 1 to 4, 4 to 4, or 4 to 1. The permutations $n \in N$ that satisfy this are $t^{yxy^{-1}} = t^{(2,5)(3,6)} = t$, and $t^x = t^{(1,4)(3,6)} = t_1^2$. We have found the presentation for the monomial progenitor $3^{*3} :_m G$ is given by $\langle x, y, t \mid x^2, y^3, (xy)^3, t^3, t^{yxy^{-1}} = t, t^x = t_1^2 \rangle$. We will now perform double coset enumeration of this monomial progenitor.

A symmetric presentation of the monomial progenitor $3^{*3} :_m A_4$ is $\langle x, y, t \mid x^2, y^3, (xy)^3, t^3, (t, xy^{-1}), t^x = t^2 \rangle$. $A_4 = \langle x, y \rangle$ of order 12, such that $x \sim (1,4)(3,6)$, $y \sim (1,2,3)(4,5,6)$, and $t \sim t_1$. We prove the above progenitor is factored by $(xyt^y)^5$. We expand the relation $(xyt^y)^5$ as follows but first, note:

$$t_1^y = t_1^{(1,2,3)(4,5,6)} = t_2$$

$$(xy)^5 = (1, 6, 5)(2, 4, 3),$$

$$(xy)^4 = (1, 5, 6)(2, 3, 4),$$

$$(xy)^3 = Id,$$

$$(xy)^2 = (1, 6, 5)(2, 4, 3), \text{ and}$$

$$(xy)^1 = (1, 5, 6)(2, 3, 4).$$

Now, the relation is expanded as $(xy)^5 t_2^{(xy)^4} t_2^{(xy)^3} t_2^{(xy)^2} t_2^{(xy)} t_2 = 1$

$$\implies (1, 6, 5)(2, 4, 3) t_2^{(1,5,6)(2,3,4)} t_2^e t_2^{(1,6,5)(2,4,3)} t_2^{(1,5,6)(2,3,4)} t_2 = 1$$

Hence, the relation is $(1, 6, 5)(2, 4, 3) t_3 t_2 t_4 t_3 t_2 = 1$.

This gives the finite homomorphic image:

$$G \cong \text{PSL}(2, 11) \cong \frac{3^{*3} \cdot_m A_4}{(1, 6, 5)(2, 4, 3) t_3 t_2 t_4 t_3 t_2 = 1}$$

We wish to find the index of N in G by the double coset enumeration algorithm of $G \cong \text{PSL}(2, 11)$ over $N \cong A_4$ as follows:

The First Double Coset $NeN = [*]$

Let NeN be denoted by $[*]$. $NeN = \{Nen \mid n \in N\} = Ne^n = N$. N is the only single coset contained in the double coset $[*]$, thus N is transitive on $\{1, 2, 3, 4, 5, 6\}$ so it has the single orbit $\{1, 2, 3, 4, 5, 6\}$. We choose 1 to be the representative from this orbit and form the coset Nt_1 such that $Nt_1 \in Nt_1N = [1]$. So, 6 symmetric generators extend to double coset $Nt_1N = \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6\}$.

The Second Double Coset $Nt_1N = [1]$

We wish to find the number of single cosets in the double coset Nt_1N . We determine the point stabilizer of 1 in N is $N^1 = \langle (2, 5)(3, 6) \rangle \cong \mathbb{Z}_2$. Now, $N^{(1)} \geq N^1 = \langle (2, 5)(3, 6) \rangle \cong \mathbb{Z}_2$. Thus, the number of right cosets in Nt_1N is

given by $\frac{|N|}{|N^{(1)}|} = \frac{12}{2} = 6$.

The orbits of $N^{(1)}$ on $\{1,2,3,4,5,6\}$ are $\{1\}, \{2, 5\}, \{3, 6\}$, and $\{4\}$. We choose 1, 2, 3, and 4 to be representatives from each orbit, and form the cosets Nt_1t_1 , Nt_1t_2 , Nt_1t_3 , and Nt_1t_4 . We now determine which double coset each coset belongs to.

$Nt_1t_1 \in [1]$:

$$Nt_1t_1 = t_1^2 = t_4 = Nt_1^{(1,4)(3,6)} \in [1].$$

Thus, 1 symmetric generator goes back to $[1]$.

$Nt_1t_4 \in [*]$:

$$Nt_1t_4 = t_1t_1^2 = t_1^3 = N.$$

Thus, 1 symmetric generator goes back to $[*]$.

$$Nt_1t_2 \in Nt_1t_2N = [12].$$

Thus, 2 symmetric generators extend to $[12]$.

$$Nt_1t_3 \in Nt_1t_3N = [13].$$

Thus, 2 symmetric generators extend to $[13]$.

The Third Double Coset $Nt_1t_2N = [12]$

We wish to find the number of single cosets in the double coset Nt_1t_2N . Now, $N^{(12)} \geq N^{12} = \langle \text{Id} \rangle$. Thus, the number of right cosets in Nt_1t_2N is given

by $\frac{|N|}{|N^{(12)}|} = \frac{|12|}{|1|} = 12$. The orbits of $N^{(12)}$ on $\{1,2,3,4,5,6\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$, and $\{6\}$. We choose 1, 2, 3, 4, 5, and 6 as representatives from each orbit and form the cosets $Nt_1t_2t_1$, $Nt_1t_2t_2$, $Nt_1t_2t_3$, $Nt_1t_2t_4$, $Nt_1t_2t_5$, and $Nt_1t_2t_6$. We determine which double coset each coset belongs to.

$$Nt_1t_2t_1 \in Nt_1t_2t_1N = [121]$$

Thus, 1 symmetric generator extends to $[121]$.

$$\underline{Nt_1t_2t_2 \in [12]:}$$

$$Nt_1t_2t_2 = t_1t_2^2 = t_1t_5 = (Nt_1t_2)^{(2,5)(3,6)} \in [12].$$

Thus, 1 symmetric generator goes back to $[12]$.

$$\underline{Nt_1t_2t_3 \in [13]:}$$

conjugate relation $(1,6,5)(2,4,3)t_3t_2t_4t_3t_2 = \text{Id}$ by $(1,3,5)(2,4,6)$ gives

$$(1,3,2)(4,6,5) = t_1t_2t_3t_1t_2$$

$$\implies t_1t_2t_3 = (1,3,2)(4,5,6)t_5t_4$$

$$\implies Nt_1t_2t_3 = Nt_5t_4 = (Nt_1t_3)^{(1,5,6)(2,3,4)} \in [13]$$

Thus, 1 symmetric generator goes back to $[13]$.

$$Nt_1t_2t_4 \in Nt_1t_2t_4N = [124].$$

Thus, 1 symmetric generator extends to $[124]$.

$$\underline{Nt_1t_2t_5 \in [1]:}$$

$$Nt_1t_2t_5 = t_1t_2t_2^2 = t_1t_2^3 = Nt_1 \in [1].$$

Thus, 1 symmetric generator goes back to [1].

$$Nt_1t_2t_6 \in Nt_1t_2t_6N = [126].$$

Thus, 1 symmetric generator extends to [126].

The Fourth Double Coset $Nt_1t_3N=[13]$

Now, $N^{(13)} \geq N^{13} = \langle \text{Id} \rangle$. Thus, the number of right cosets in Nt_1t_3N is given by $\frac{|N|}{|N^{(13)}|} = \frac{12}{1} = 12$. The orbits of $N^{(13)}$ on $\{1,2,3,4,5,6\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$, and $\{6\}$. We choose a representative from each orbit and form the cosets $Nt_1t_3t_1, Nt_1t_3t_2, Nt_1t_3t_3, Nt_1t_3t_4, Nt_1t_3t_5$, and $Nt_1t_3t_6$. We determine which double coset each coset belongs to.

$$\underline{Nt_1t_3t_1 \in [121]}:$$

$$\text{Let } 1 \ 3 \ 1 = 1 \ 6 \ \underline{6} \ \underline{1}$$

$$\text{conjugate relation by } (1,5,6)(2,3,4) \text{ gives } 6 \ 1 = (1,6,5)(2,4,3) \ 4 \ 3 \ 2$$

$$\text{Now, } 1 \ 3 \ 1 = 1 \ 6 \ (1,6,5)(2,4,3) \ 4 \ 3 \ 2 = (1,6,5)(2,4,3) \ \underline{6} \ \underline{5} \ \underline{4} \ 3 \ 2$$

$$\text{conjugate relation by } (1,2,6)(3,4,5) \text{ gives}$$

$$6 \ 5 \ 4 = 1 \ (1,2,3)(4,5,6) \ 3 = (1,2,3)(4,5,6) \ 2 \ 3$$

$$\text{Now, } 1 \ 3 \ 1 = (1,4)(2,5) \ 2 \ \underline{3} \ \underline{3} \ 2 = (1,4)(2,5) \ 2 \ 6 \ 2 = (121)^{(1,2,6)(3,4,5)}$$

$$\text{So, } Nt_1t_3t_1 = Nt_1t_2t_1 \in [121].$$

Thus, 1 symmetric generator goes back to [121].

$$\underline{Nt_1t_3t_2 \in [126]}:$$

$$\text{Let } t_1t_3t_2 = t_4 \ \underline{t_4} \ t_3 \ t_2$$

re-write relation as $4\ 3\ 2 = (1,5,6)(2,3,4)\ 6\ 1$

Now, $1\ 3\ 2 = 4\ (1,5,6)(2,3,4)\ 6\ 1 = (1,6,5)(2,3,4)\ 2\ 6\ 1$

$$= (1,6,5)(2,3,4)\ (t_1 t_2 t_6)^{(1,2,6)(3,4,5)}$$

$$\implies Nt_1 t_3 t_2 \in Nt_1 t_2 t_6 N = [1, 2, 6]$$

Thus, 1 symmetric generator extends to $[126]$.

$Nt_1 t_3 t_3 \in [13]$:

$$Nt_1 t_3 t_3 = t_1 t_3^2 = t_1 t_6 = (t_1 t_3)^{(2,5)(3,6)}$$

So, $Nt_1 t_3 t_3 = Nt_1 t_6 \in [13]$

Thus, 1 symmetric generator goes back to $[13]$.

$$Nt_1 t_3 t_4 \in Nt_1 t_3 t_4 N = [134].$$

Thus, 1 symmetric generator extends to $[134]$.

$Nt_1 t_3 t_5 \in [12]$:

Conjugate relation by $(1,2,3)(4,5,6)$

$$\implies (1,3,5)(2,4,6)t_1 t_3 t_5 t_1 t_3 = Id$$

$$\implies (1,3,5)(2,4,6)t_1 t_3 t_5 = t_6 t_4$$

$$\implies Nt_1 t_3 t_5 = Nt_6 t_4 = (Nt_1 t_2)^{(1,6,5)(2,4,3)} \in [12].$$

Thus, 1 symmetric generator extends to $[12]$.

$Nt_1 t_3 t_6 \in [1]$:

$$Nt_1 t_3 t_6 = Nt_1 t_3 t_3^2 = Nt_1 t_3^3 = Nt_1 \in [1].$$

Thus, 1 symmetric generator goes back to $[1]$.

The Fifth Double Coset $Nt_1t_2t_1N = [121]$

We wish to find the number of single cosets in the double coset $Nt_1t_2t_1N$. Now, $N^{(121)} \geq N^{121} = \langle \text{Id} \rangle$. Thus, the number of right cosets in $Nt_1t_2t_1N$ is given by $\frac{|N|}{|N^{(121)}|} = \frac{12}{1} = 12$. The orbits of $N^{(121)}$ on $\{1,2,3,4,5,6\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$, and $\{6\}$. We choose a representative from each orbit and form the cosets $Nt_1t_2t_1t_1$, $Nt_1t_2t_1t_2$, $Nt_1t_2t_1t_3$, $Nt_1t_2t_1t_4$, $Nt_1t_2t_1t_5$, and $Nt_1t_2t_1t_6$. We determine which double coset each coset belongs to.

We prove a lemma that will help us prove some relations, later.

Lemma: $121 = 656$

Proof:

$$121323 = 1\ 2\ 1\ 3\ 2\ 3\ 1\ 2\ 3\ 6\ 5\ 4 \text{ since } 123654 = \text{Id}$$

$$= 1\ 2\ 1\ 3\ \underline{2\ 3\ 1\ 2\ 3}\ 6\ 5\ 4$$

$$\text{conjugate relation by } (2,5)(3,6) \text{ gives } 23123 = (1, 3, 2)(4, 6, 5)$$

$$= 1\ 2\ 1\ 3\ (1, 3, 2)(4, 6, 5)654$$

$$= (1, 3, 2)(4, 6, 5)\ 31326\underline{54}$$

$$\text{conjugate relation by } (1,3,5)(2,4,6) \text{ gives } 54 = (1,2,3)(4,5,6)123$$

$$= (1, 3, 2)(4, 6, 5)\ 31326(1,2,3)(4,5,6)123$$

$$= (1, 3, 2)(4, 6, 5)(1,2,3)(4,5,6)12134123$$

$$= \underline{12134123}$$

conjugating relation by $(1, 2, 3)(4, 5, 6)$ gives $13=(1,5,3)(2,6,4)642$

$$= 12(1, 5, 3)(2, 6, 4)6424123$$

$$= (1, 5, 3)(2, 6, 4) \underline{5664241}23$$

$$= (1, 5, 3)(2, 6, 4) \underline{53422}3$$

$$= (1, 5, 3)(2, 6, 4) \underline{53453}$$

conjugating relation by $(1, 4)(3, 6)$ we get $(1,6,2)(3,5,4)=53453$

$$= (1, 5, 3)(2, 6, 4)(1,6,2)(3,5,4)$$

$$= (1,4)(3,6)$$

Now, $121323 = (1,4)(3,6)$.

So $121 = (1,4)(3,6) \ 656$.

Thus, $121 = 656$. \square

$Nt_1t_2t_1t_1 \in [124]$:

$$Nt_1t_2t_1t_1 = Nt_1t_2t_1^2 = Nt_1t_2t_4 \in [124].$$

Thus, 1 symmetric generator goes to $[124]$.

$Nt_1t_2t_1t_2 \in [121]$:

$$1212 = 121\underline{23}6$$

conjugate relation by $(1,3,5)(2,4,6)$ gives is $(1,3,2)(4,6,5)54=123$

$$= 12(1, 3, 2)(4, 6, 5)546$$

$$= (1, 3, 2)(4, 6, 5) \underline{31}546 \ (3 = 66)$$

$$= (1, 3, 2)(4, 6, 5)\underline{6615}46$$

conjugate relation by $(1,5,6)(2,3,4)$ gives is $(1,6,5)(2,4,3)43=615$

$$= (1, 3, 2)(4, 6, 5) 6(1, 6, 5)(2, 4, 3)4346$$

$$= (1, 2, 6)(3, 4, 5) \underline{54}346$$

conjugate relation by $(1,3,5)(2,4,6)$ gives is $54=(1,2,3)(4,5,6)123$

$$= (1, 2, 6)(3, 4, 5)(1,2,3)(4,5,6)123\underline{34}6$$

$$= (1,3,5)(2,4,6) \underline{126}46 \text{ } (Nt_1t_2t_6 = Nt_2t_3t_4 \text{ - see below})$$

$$= (1,3,5)(2,4,6) \underline{2344}6$$

$$= (1,3,5)(2,4,6) \underline{231}6$$

conjugate relation by $(2,5)(3,6)$ gives is $123=(1,2,3)(4,5,6)65$

$$= (1, 3, 5)(2, 4, 6)(1,2,3)(4,5,6)656.$$

Thus $1212= (2,5)(3,6) 656$.

$$\implies Nt_1t_2t_1t_2 = Nt_6t_5t_6 = Nt_1t_2t_1 \text{ (Lemma)} \in Nt_1t_2t_1N = [121].$$

Thus, 1 symmetric generator goes back to $[121]$.

$$\underline{Nt_1t_2t_1t_3 \in [13]}:$$

$$Nt_1t_2t_1 = Nt_6t_5t_6 \text{ (Lemma)}$$

multiply by t_3

$$Nt_1t_2t_1t_3 = t_6t_5t_6t_3 = t_6t_5t_3t_3t_3 = t_6t_5 = (t_1t_3)^{(1,6,2)(3,5,4)}$$

So, $Nt_1t_2t_1t_3 = Nt_1t_3 \in Nt_1t_3N = [13]$.

Thus, 1 symmetric generator goes back to [13].

$Nt_1t_2t_1t_4 \in [12]$:

$$Nt_1t_2t_1t_4 = t_1t_2t_1t_1^2 = t_1t_2t_1^3 = Nt_1t_2 \in [12].$$

Thus, 1 symmetric generator goes back to [12].

$Nt_1t_2t_1t_5 \in [121]$:

$$Nt_1t_2t_1t_5 = t_1t_2t_1t_2t_2 \text{ (1212 = 656 see above)} = Nt_6t_5t_6t_2$$

$$\text{Thus, } Nt_1t_2t_1t_5 = Nt_6t_5t_6t_2$$

$$\implies Nt_1t_2t_1t_5t_5 = Nt_6t_5t_6$$

$$\implies Nt_1t_2t_1t_2 = Nt_6t_5t_6 = Nt_1t_2t_1$$

$$\text{Hence, } Nt_1t_2t_1t_2 = Nt_1t_2t_1 \in [121].$$

Thus, 1 symmetric generator goes back to [121].

$Nt_1t_2t_1t_6 \in [134]$:

$$\text{Let } 1216 = \underline{1216}$$

$$\text{conjugate relation by } (1,5,3)(2,6,4) \text{ gives } 216 = (1,2,6)(3,4,5)45$$

$$= 1(1,2,6)(3,4,5)45$$

$$= (1,2,6)(3,4,5)245 \text{ (134 = 245 see below)}$$

$$\text{Now, } 1216 = (1,2,6)(3,4,5) \text{ 245.}$$

$$\text{Thus, } Nt_1t_2t_1t_6 \in = Nt_1t_3t_4.$$

Hence, 1 symmetric generator goes to [134].

The Sixth Double Coset $Nt_1t_2t_4N = [124]$

We wish to find the number of single cosets in the double coset $Nt_1t_2t_4N$.

Lemma 124 = 532:

Proof:

124562

re-write relation by $(1,6,5)(2,4,3)324=56$

$$= 124(1, 6, 5)(2, 4, 3)3242$$

$$= (1, 6, 5)(2, 4, 3) \ 64\underline{33}242$$

$$= (1, 6, 5)(2, 4, 3) \ 646242 = (1, 6, 5)(2, 4, 3) \ 64\underline{62}42$$

conjugating relation by $(1,4)(3,6)$ gives $62=(1, 2, 6)(3, 4, 5)534$

$$=(1,6,5)(2,4,3)64(1, 2, 6)(3, 4, 5)53442$$

$$= (2, 5)(3, 6) \ 1\underline{55}3\underline{44}2$$

$$= (2, 5)(3, 6) \ \underline{12312}$$

conjugating relation by $(1,3,5)(2,4,6)$ gives $(1, 3, 2)(4, 6, 5)=12312$

$$= (2,5)(3,6)(1, 3, 5)(2, 4, 6)$$

$$= (1,3,2)(4,6,5)$$

$$\implies 124562 = (1,3,2)(4,6,5)$$

$$\implies 124 = (1,3,2)(4,6,5) \ 532$$

So, $124 = 532$. \square

Now $N(t_1t_2t_4)^{(1,5,6)(2,3,4)} = Nt_5t_3t_2 = Nt_1t_2t_4$ implies $(1,5,6)(2,3,4)$ belongs to the coset stabilizer $N^{(124)} \geq < N^{124}, (1,5,6)(2,3,4) > \cong \mathbb{Z}_3$. Since $[124]^{N^{(124)}}$ gives

$\{[124], [532], [643]\}$, we have $124 \sim 532 \sim 643$. Thus, the number of cosets in $Nt_1t_2t_4N$ is given by $\frac{|N|}{|N^{(124)}|} = \frac{12}{3} = 4$.

The orbits of $N^{(124)}$ on $\{1,2,3,4,5,6\}$ are $\{1, 5,6\}$ and $\{2,3,4\}$. We choose a representative from each orbit and form the cosets $Nt_1t_2t_4t_1$, and $Nt_1t_2t_4t_4$. We now determine which double coset each coset belongs to.

$Nt_1t_2t_4t_1 \in [12]$:

$$Nt_1t_2t_4t_1 = t_1t_2t_1^2t_1 = t_1t_2t_1^3 = Nt_1t_2$$

thus, 3 symmetric generator goes back to $[12]$.

$Nt_1t_2t_4t_4 \in [121]$:

$$Nt_1t_2t_4t_4 = t_1t_2t_1^2t_1^2 = t_1t_2t_1^4 = t_1t_2t_1^3t_1 = Nt_1t_2t_1$$

Thus, 3 symmetric generators go back to $[121]$.

No new double cosets are formed from this double coset so this coset is closed under coset multiplication.

The Seventh Double Coset $Nt_1t_2t_6N = [126]$

We wish to find the number of single cosets in the double coset $Nt_1t_2t_6N$.

Lemma: $126 = 234$

Proof:

1 2 6 1 6 5

conjugate relation by $(2,5)(3,6)$ gives $65=(1,2,3)(4,5,6)2\ 3\ 1$

$$= 1 \ 2 \ 6 \ 1 \ (1,2,3)(4,5,6)2 \ 3 \ 1$$

$$= (1,2,3)(4,5,6) \ 234\underline{22}31$$

$$= (1,2,3)(4,5,6) \ 234\underline{53}1$$

conjugate relation by $(1,5,3)(2,6,4)$ gives $3453=(1, 6, 2)(3, 5, 4)1$

$$= (1,2,3)(4,5,6)2(1, 6, 2)(3, 5, 4)11$$

$$= (2, 5)(3, 6) \ \underline{111} \ (111 = \text{Id})$$

$$= (2, 5)(3, 6).$$

$$\text{So, } 126165 = (2, 5)(3, 6).$$

$$\implies 126 = (2,5)(3,6)234.$$

Thus, $126 = 234$.

Now $N(t_1t_2t_6)^{(1,2,3)(4,5,6)} = Nt_2t_3t_4 = Nt_1t_2t_6$ implies $(1, 2, 3)(4, 5, 6)$ belongs to $N^{(126)} \geq < N^{126}, (1, 2, 3)(4, 5, 6) > \cong \mathbb{Z}_3$. Since $[126]^{N^{(126)}}$ gives $\{[126], [234], [315]\}$, we have $126 \sim 234 \sim 315$. Thus, the number of cosets in $Nt_1t_2t_6N$ is given by $\frac{|N|}{|N^{(126)}|} = \frac{12}{3} = 4$. The orbits of $N^{(126)}$ on $\{1, 2, 3, 4, 5, 6\}$ are $\{1, 2, 3\}$ and $\{4, 5, 6\}$. We choose 3 and 6 to be the representatives from each orbit and form the cosets $Nt_1t_2t_6t_3$ and $Nt_1t_2t_6t_6$. We now determine which double coset each single coset belongs to.

$$\underline{Nt_1t_2t_6t_3 \in [12]}:$$

$$Nt_1t_2t_6t_3 = t_1t_2t_3^2t_3 = t_1t_2t_3^3 = Nt_1t_2$$

$$\text{So, } Nt_1t_2t_6t_3 = Nt_1t_2t_6t_3$$

Thus, 3 symmetric generator goes back to $[12]$.

$Nt_1t_2t_6t_6 \in [13]$:

$$Nt_1t_2t_6t_6 = t_1t_2t_3^2t_3^2 = t_1t_2t_3^4 = t_1t_2t_3^3t_3 = t_1t_2t_3$$

So, $Nt_1t_2t_6t_6 \in [13]$

Thus, 3 symmetric generators go back to $[13]$.

No new double cosets are formed from this double coset so this coset is closed under coset multiplication.

The Eighth Double Coset $Nt_1t_3t_4N = [134]$

We wish to find the number of single cosets in the double coset $Nt_1t_3t_4N$.

Lemma: $134 = 245$

Proof:

$$134 = \underline{134}$$

conjugate relation by $(1,5,3)(2,6,4)$ gives $34 = (1, 6, 2)(3, 5, 4)162$

$$= 1(1, 6, 2)(3, 5, 4)162$$

$$= (1, 6, 2)(3, 5, 4) \underline{61}62$$

conjugate relation by $(1,5,6)(2,3,4)$ gives $61 = (1, 6, 5)(2, 4, 3)432$

$$= (1, 6, 2)(3, 5, 4)(1,6,5)(2,4,3)43262$$

$$= (1, 5, 3)(2, 6, 4) 432\underline{6}2$$

$$= (1, 5, 3)(2, 6, 4) 432332$$

$$= (1, 5, 3)(2, 6, 4) 43233\underline{2}$$

conjugate relation by Id gives $32=(1, 5, 6)(2, 3, 4)561$

$$= (1, 5, 3)(2, 6, 4) 4323(1, 5, 6)(2, 3, 4)561$$

$$= (1, 6, 2)(3, 5, 4) 24\overline{345}61$$

conjugate relation by $(1,5,3)(2,6,4)$ gives $345=(1, 6, 2)(3, 5, 4)16$

$$= (1, 6, 2)(3, 5, 4) 24(1, 6, 2)(3, 5, 4)\overline{166}1$$

$$= (1, 2, 6)(3, 4, 5) 13131.$$

So $134 = (1, 2, 6)(3, 4, 5) 13131$.

Also, $\overline{245}$

conjugate relation by $(1,6,5)(2,4,3)$ gives $24=(1, 5, 6)(2, 3, 4)156$

$$= (1, 5, 6)(2, 3, 4)1565$$

$$= (1, 5, 6)(2, 3, 4) 15\overline{65}$$

conjugate relation by $(1, 5)(2, 6)$ gives $65=(1, 2, 3)(4, 5, 6)231$

$$= (1, 5, 6)(2, 3, 4) 15(1, 2, 3)(4, 5, 6)231$$

$$= (1, 6, 2)(3, 5, 4) 26231.$$

Then $245 = (1, 6, 2)(3, 5, 4) 26231$

Now, $121 = (1, 2, 6)(3, 4, 5) 656$ (Lemma).

Then $121^{(1,2,6)(3,4,5)} = (1, 2, 6)(3, 4, 5) 656^{(1,2,6)(3,4,5)}$ (Lemma).

So, $262 = (1, 2, 6)(3, 4, 5) 131$

Thus, $245 = (1, 2, 6)(3, 4, 5) 13131$

Therefore, $134 = (1, 2, 6)(3, 4, 5) 13131 = 245$.

Thus, $134 = 245$. \square

Now, $N(t_1 t_3 t_4)^{(1,2,6)(3,4,5)} = N t_2 t_4 t_5 = N t_1 t_3 t_4$ implies $(1, 2, 6)(3, 4, 5)$ belongs to $N^{(134)} \geq < N^{134}, (1, 2, 6)(3, 4, 5) > \cong \mathbb{Z}_3$. Since $[134]^{N^{(134)}}$ gives $\{[134], [245], [653]\}$, we have $134 \sim 245 \sim 653$. Thus, the number of cosets in $N t_1 t_3 t_4 N$ is given by $\frac{|N|}{|N^{(134)}|} = \frac{12}{3} = 4$. The orbits of $N^{(134)}$ on $\{1, 2, 3, 4, 5, 6\}$ are $\{1, 2, 6\}$ and $\{3, 4, 5\}$. We choose a representative from each orbit and form the cosets $N t_1 t_3 t_4 t_1$, and $N t_1 t_3 t_4 t_4$. We now determine which double coset each coset belongs to.

$N t_1 t_3 t_4 t_1 \in [13]$:

$$N t_1 t_3 t_4 t_1 = t_1 t_3 t_1^2 t_1 = t_1 t_3 t_1^3 = N t_1 t_3$$

So $N t_1 t_3 t_4 t_1 = N t_1 t_3 \in [13]$.

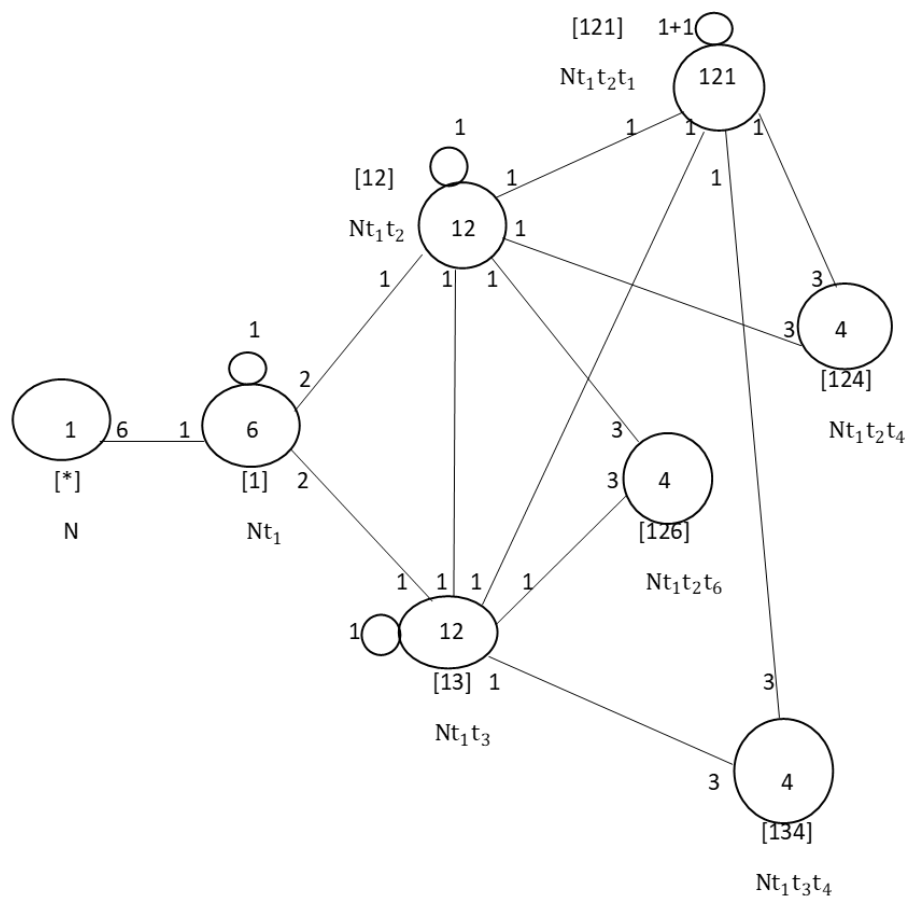
Thus, 3 symmetric generators go back to $[13]$.

$N t_1 t_3 t_4 t_4 \in [121]$:

$$N t_1 t_3 t_4 t_4 = t_1 t_3 t_1^2 t_1^2 = t_1 t_3 t_1^4 = t_1 t_3 t_1^3 t_1 = t_1 t_3 t_1$$

So, $N t_1 t_3 t_4 t_4 = N t_1 t_3 t_1 \in [121]$ (see previously)

Thus, 3 symmetric generators go back to $[121]$. We have completed the double coset enumeration.

Figure 8.1: Cayley Graph of DCE $\text{PSL}(2,11)$ over A_4

8.2 DCE of $\text{PSL}(2,11)$ over Maximal Subgroup A_5 and A_4

We begin by factoring the progenitor $3^{*3} :_m A_4$ by the relation $(xyt^y)^5$.

This gives the finite homomorphic image:

$\text{PSL}(2,11) \cong \frac{3^{*3} :_m A_4}{(xyt^y)^5}$, where $x \sim (1,4)(3,6)$, $y \sim (1,2,3)(4,5,6)$, and $t \sim t_1$.

Let H be a subgroup generated by $N = A_4$ and $xyx^{-1}t^xt(xy^2)t$. Then $H = \langle N, (xyx^{-1})t_4t_6t_1 \rangle$, such that H is maximal and $|H| = 60$. We now proceed with the double coset enumeration of G over H and N .

The First Double Coset $HtN = [*]$

The number of single cosets contained in $[*]$ is determined by $\frac{|N|}{|H|}$. However, the only single coset contained in $[*]$ is H . Thus, the number of single cosets of H in N is $\frac{12}{12} = 1$. The orbits of N on $\{1, 2, 3, 4, 5, 6\}$ is $\{1, 2, 3, 4, 5, 6\}$. We choose 1 to be the representative from this orbit and form the new double coset Ht_1N . Thus, 6 symmetric generators will move forward to $Ht_1N = [1]$.

The Second Double Coset $Ht_1N = [1]$

The number of cosets contained in this double coset is found by $\frac{|N|}{|N^{(1)}|}$, where $N^{(1)}$ is the coset stabilizer which is always equal to or greater than the point-stabilizer of 1, denoted by N^1 . Now $N^{(1)} \geq N^1 = \langle (2,5)(3,6) \rangle \cong \mathbb{Z}_2$.

Thus, the number of single cosets of H in N are $\frac{12}{2} = 6$. The orbits of $N^{(1)}$ on $\{1, 2, 3, 4, 5, 6\}$ are $\{1\}$, $\{4\}$, $\{2, 5\}$, and $\{3, 6\}$. We choose 1, 2, 3, and 4 to be the representatives from each orbit and form the cosets Ht_1t_1 , Ht_1t_2 , Ht_1t_3 , and Ht_1t_4 . We now determine which double coset each single coset belongs to.

$Ht_1t_1 \in [1]$:

$$Ht_1t_1 = t_1^2 = t_4 = Ht_4 \in Ht_1N = [1].$$

Thus, 1 symmetric generator goes back to $[1]$.

$$Ht_1t_2 \in Ht_1t_2N = [12].$$

Thus, 2 symmetric generators extend to $[12]$.

$Ht_1t_3 \in [1]$:

$$H = Ht_4t_6t_1.$$

multiply by t_1t_3

$$Ht_1t_3 = Ht_4t_6t_1t_1t_3$$

$$Ht_1t_3 = H(t_4t_6t_1)^{(1,4)(2,5)}t_1t_3, \text{ since } H = Ht_1t_6t_4 = H(t_4t_6t_1)^{(1,4)(2,5)}$$

$$Ht_1t_3 = Ht_1t_6t_4t_1t_3$$

$$Ht_1t_3 = Ht_1t_6t_3$$

$$Ht_1t_3 = Ht_1$$

Thus, 2 symmetric generators go back to $[1]$.

$Ht_1t_4 \in [*]$:

$$Ht_1t_4 = Ht_1t_1^2 = Ht_1^3 = H.$$

Thus, 1 symmetric generator goes back to $[\ast]$.

The Third Double Coset $Ht_1t_2N = [12]$

We wish to find the number of single cosets of H in the double coset Ht_1t_2N .

Lemma: $Ht_1t_2 = Ht_5t_6$

Proof:

The relation gives $(1, 3, 2)(4, 6, 5)54 = 123$

So $t_5t_4 = (1, 2, 3)(4, 5, 6)t_1t_2t_3$

Thus, $H(1, 2, 3)(4, 5, 6)t_1t_2t_3 = Ht_5t_4$

So, $Ht_1t_2t_3 = Ht_5t_4$

Then, $Ht_1t_2t_3 = Ht_5t_4t_2t_5$ ($25 = \text{Id}$)

But $H = Ht_4t_6t_1$ (by relation $(2,5)(3,6)t_4t_6t_1$)

$$\implies H = H(t_4t_6t_1)^{(1,2,3)(4,5,6)} = Ht_5t_4t_2$$

$$\implies H = Ht_5t_4t_2$$

$$\implies Ht_5 = Ht_5t_4$$

So, $Ht_5t_4t_2t_5 = Ht_5$ Thus, $Ht_1t_2t_3 = Ht_5t_4t_2t_5 = Ht_5$; that is, $Ht_1t_2t_3 = Ht_5$.

So, $Ht_1t_2 = Ht_5t_6$. \square

The number of cosets contained in this double coset is found by $\frac{|N|}{|N^{(12)}|}$ where $N^{(12)}$ is the coset stabilizer which is always equal to or greater than the point-

stabilizer of 1 and 2, denoted by N^{12} . Now, $H(t_1 t_2)^{(1,5,3)(2,6,4)} = H t_5 t_6 = H t_1 t_2$ implies $(1, 5, 3)(2, 6, 4)$ belongs to $N^{(12)} \geq < N^{12}, (1, 5, 3)(2, 6, 4) > \cong \mathbb{Z}_3$. Since $[12]^{N^{(12)}}$ gives $\{[12], [56], [34]\}$, we have $12 \sim 34 \sim 56$. Thus, the number of cosets of H in N are $\frac{12}{3} = 4$.

The orbits of $N^{(12)}$ on $\{1, 2, 3, 4, 5, 6\}$ are $\{1, 5, 3\}$ and $\{2, 6, 4\}$. We choose 5 and 2 to be the representatives from each orbit and form the cosets $H t_1 t_2 t_5$ and $H t_1 t_2 t_2$. We now determine which double coset each single coset belongs to.

$$H t_1 t_2 t_5 = t_1 t_2 t_2^2 = H t_1 \in [1].$$

Thus, 3 symmetric generators go back to $[1]$.

$$H t_1 t_2 t_2 = t_1 t_2^2 = t_1 t_5 = H t_1 t_5 = H(t_1 t_2)^{(2,5)(3,6)}$$

So, $H t_1 t_2 t_2 \in [12]$.

Thus, 3 symmetric generators go back to $[12]$.

Since, no new cosets are formed this coset is closed under coset multiplication and we have completed the double coset enumeration for G over H and N, as depicted in the Cayley graph.

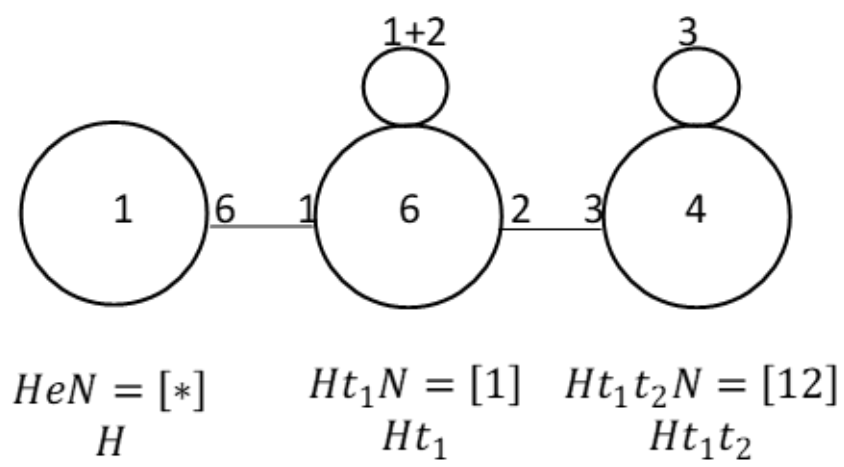


Figure 8.2: Cayley Graph of $PSL(2,11)$ over Maximal Subgroup A_5 and A_4

Chapter 9

Related Topics

9.1 Wreath Products

Given two groups H and K . The wreath product of H by K , denoted $H \wr K$ is a semi-direct product composed of as many copies of H as the number of letters on which the permutation group K acts on. We define the wreath product below.

Definition 5.1.1. (Wreath Product) *Let X and Y be non-empty sets. Let H be a permutation group on X and K on Y . Let $Z = X \times Y$, then the **wreath product** is a semi-direct product of X and Y . We define a permutation group on Z such that we let $\gamma \in H$, and define a permutation of $\gamma(y)$ of Z by*

$$\gamma(y) = \begin{cases} (x, y) \mapsto (x\gamma, y), & \text{if } y_1 = y \\ (x, y_1) \mapsto (x, y_1), & y \neq y_1 \end{cases}$$

Also, for $k \in K$, define $k^(x, y) = (x, (y)k)$; such that $B = \bigcup_{y \in Y} H(y)$ is a direct product of the group generated by the y 's. Thus, $G = B:k^*$ is called a wreath*

product of H and K , where H is normal subgroup, denoted by $H \wr K$. [Rot95]

We look at a few examples to help us better understand how wreath products are generated.

9.1.1 Example: $S_3 \wr 2^2$

Let $H = \langle (1, 2, 3), (1, 2) \rangle \cong S_3$ and $K = \langle (4, 5), (6, 7) \rangle \cong 2^2$ be permutation groups on $X = \{ 1, 2, 3 \}$ and $Y = \{ 4, 5, 6, 7 \}$, respectively. We Construct, by hand, permutation generators of the wreath product, $H \wr K$, of H and K . Let $Z = X \times Y = \{ (1,4), (1,5), (1,6), (1,7), (2,4), (2,5), (2,6), (2,7), (3,4), (3,5), (3,6), (3,7) \}$. We consider the labeling for the elements of Z :

Table 9.1: Labeling of Z

Labeling	Element
8	(1,4)
9	(1,5)
10	(1,6)
11	(1,7)
12	(2,4)
13	(2,5)
14	(2,6)
15	(2,7)
16	(3,4)
17	(3,5)
18	(3,6)
19	(3,7)

Using wreath product definition we let $\gamma_1 = (1,2,3)$, $\gamma_2 = (1,2) \in H$ and $y \in Y$ and compute $\gamma_1(4)$, $\gamma_1(5)$, $\gamma_1(6)$, $\gamma_1(7)$, $\gamma_2(4)$, $\gamma_2(5)$, $\gamma_2(6)$, and $\gamma_2(7)$. We compute the action of $\gamma_1=(1,2,3)$ on K . Notice by definition this application of γ will only change elements which contain 1, 2, and 3 in the x-coordinate and 4 in the y-

coordinate.

Table 9.2: Labeling of $\gamma_1(4)$

Labeling	Element				Element	Labeling
8	(1,4)	\longrightarrow	$(\gamma(1),4)$	\longrightarrow	(2,4)	12
9	(1,5)	\longrightarrow	$(\gamma(1),5)$	\longrightarrow	(1,5)	9
10	(1,6)	\longrightarrow	$(\gamma(1),6)$	\longrightarrow	(1,6)	10
11	(1,7)	\longrightarrow	$(\gamma(1),7)$	\longrightarrow	(1,7)	11
12	(2,4)	\longrightarrow	$(\gamma(2),4)$	\longrightarrow	(3,4)	16
13	(2,5)	\longrightarrow	$(\gamma(2),5)$	\longrightarrow	(2,5)	13
14	(2,6)	\longrightarrow	$(\gamma(2),6)$	\longrightarrow	(2,6)	14
15	(2,7)	\longrightarrow	$(\gamma(2),7)$	\longrightarrow	(2,7)	15
16	(3,4)	\longrightarrow	$(\gamma(3),4)$	\longrightarrow	(1,4)	8
17	(3,5)	\longrightarrow	$(\gamma(3),5)$	\longrightarrow	(3,5)	17
18	(3,6)	\longrightarrow	$(\gamma(3),6)$	\longrightarrow	(3,6)	18
19	(3,7)	\longrightarrow	$(\gamma(3),7)$	\longrightarrow	(3,7)	19

Table 9.3: Labeling of $\gamma_1(5)$

Labeling	Element				Element	Labeling
8	(1,4)	\longrightarrow	$(\gamma(1),4)$	\longrightarrow	(1,4)	8
9	(1,5)	\longrightarrow	$(\gamma(1),5)$	\longrightarrow	(2,5)	13
10	(1,6)	\longrightarrow	$(\gamma(1),6)$	\longrightarrow	(1,6)	10
11	(1,7)	\longrightarrow	$(\gamma(1),7)$	\longrightarrow	(1,7)	11
12	(2,4)	\longrightarrow	$(\gamma(2),4)$	\longrightarrow	(2,4)	12
13	(2,5)	\longrightarrow	$(\gamma(2),5)$	\longrightarrow	(3,5)	17
14	(2,6)	\longrightarrow	$(\gamma(2),6)$	\longrightarrow	(2,6)	14
15	(2,7)	\longrightarrow	$(\gamma(2),7)$	\longrightarrow	(2,7)	15
16	(3,4)	\longrightarrow	$(\gamma(3),4)$	\longrightarrow	(3,4)	16
17	(3,5)	\longrightarrow	$(\gamma(3),5)$	\longrightarrow	(1,5)	9
18	(3,6)	\longrightarrow	$(\gamma(3),6)$	\longrightarrow	(3,6)	18
19	(3,7)	\longrightarrow	$(\gamma(3),7)$	\longrightarrow	(3,7)	19

Table 9.4: Labeling of $\gamma_1(6)$

Labeling	Element				Element	Labeling
8	(1,4)	\longrightarrow	$(\gamma(1),4)$	\longrightarrow	(1,4)	8
9	(1,5)	\longrightarrow	$(\gamma(1),5)$	\longrightarrow	(1,5)	9
10	(1,6)	\longrightarrow	$(\gamma(1),6)$	\longrightarrow	(2,6)	14
11	(1,7)	\longrightarrow	$(\gamma(1),7)$	\longrightarrow	(1,7)	11
12	(2,4)	\longrightarrow	$(\gamma(2),4)$	\longrightarrow	(2,4)	12
13	(2,5)	\longrightarrow	$(\gamma(2),5)$	\longrightarrow	(2,5)	13
14	(2,6)	\longrightarrow	$(\gamma(2),6)$	\longrightarrow	(3,6)	18
15	(2,7)	\longrightarrow	$(\gamma(2),7)$	\longrightarrow	(2,7)	15
16	(3,4)	\longrightarrow	$(\gamma(3),4)$	\longrightarrow	(3,4)	16
17	(3,5)	\longrightarrow	$(\gamma(3),5)$	\longrightarrow	(3,5)	17
18	(3,6)	\longrightarrow	$(\gamma(3),6)$	\longrightarrow	(1,6)	10
19	(3,7)	\longrightarrow	$(\gamma(3),7)$	\longrightarrow	(3,7)	19

Table 9.5: Labeling of $\gamma_1(7)$

Labeling	Element				Element	Labeling
8	(1,4)	\longrightarrow	$(\gamma(1),4)$	\longrightarrow	(1,4)	8
9	(1,5)	\longrightarrow	$(\gamma(1),5)$	\longrightarrow	(1,5)	9
10	(1,6)	\longrightarrow	$(\gamma(1),6)$	\longrightarrow	(1,6)	10
11	(1,7)	\longrightarrow	$(\gamma(1),7)$	\longrightarrow	(2,7)	15
12	(2,4)	\longrightarrow	$(\gamma(2),4)$	\longrightarrow	(2,4)	12
13	(2,5)	\longrightarrow	$(\gamma(2),5)$	\longrightarrow	(2,5)	13
14	(2,6)	\longrightarrow	$(\gamma(2),6)$	\longrightarrow	(2,6)	14
15	(2,7)	\longrightarrow	$(\gamma(2),7)$	\longrightarrow	(3,7)	19
16	(3,4)	\longrightarrow	$(\gamma(3),4)$	\longrightarrow	(3,4)	16
17	(3,5)	\longrightarrow	$(\gamma(3),5)$	\longrightarrow	(3,5)	17
18	(3,6)	\longrightarrow	$(\gamma(3),6)$	\longrightarrow	(3,6)	18
19	(3,7)	\longrightarrow	$(\gamma(3),7)$	\longrightarrow	(1,7)	11

The labeling gives yields $\gamma_1(4) = ((1,4),(2,4),(3,4)) = (8,12,16)$, $\gamma_1(5) = (9,13,17)$, $\gamma_1(6) = (10,14,18)$, and $\gamma_1(7) = (11,15,19)$. Now, we compute the action of $\gamma_2 = (1,2)$ on K , such that we compute $\gamma_2(4)$, $\gamma_2(5)$, $\gamma_2(6)$, and $\gamma_2(7)$. As,

previously by definition of wreath product this action will only change the the elements of Z which have 1 and 2 in the x-coordinate.

Table 9.6: Labeling of $\gamma_2(4)$

Labeling	Element				Element	Labeling
8	(1,4)	\longrightarrow	$(\gamma(1),4)$	\longrightarrow	(2,4)	12
9	(1,5)	\longrightarrow	$(\gamma(1),5)$	\longrightarrow	(1,5)	9
10	(1,6)	\longrightarrow	$(\gamma(1),6)$	\longrightarrow	(1,6)	10
11	(1,7)	\longrightarrow	$(\gamma(1),7)$	\longrightarrow	(1,7)	11
12	(2,4)	\longrightarrow	$(\gamma(2),4)$	\longrightarrow	(1,4)	8
13	(2,5)	\longrightarrow	$(\gamma(2),5)$	\longrightarrow	(2,5)	13
14	(2,6)	\longrightarrow	$(\gamma(2),6)$	\longrightarrow	(2,6)	14
15	(2,7)	\longrightarrow	$(\gamma(2),7)$	\longrightarrow	(2,7)	15
16	(3,4)	\longrightarrow	$(\gamma(3),4)$	\longrightarrow	(3,4)	16
17	(3,5)	\longrightarrow	$(\gamma(3),5)$	\longrightarrow	(3,5)	17
18	(3,6)	\longrightarrow	$(\gamma(3),6)$	\longrightarrow	(3,6)	18
19	(3,7)	\longrightarrow	$(\gamma(3),7)$	\longrightarrow	(3,7)	19

Table 9.7: Labeling of $\gamma_2(5)$

Labeling	Element				Element	Labeling
8	(1,4)	\longrightarrow	$(\gamma(1),4)$	\longrightarrow	(1,4)	8
9	(1,5)	\longrightarrow	$(\gamma(1),5)$	\longrightarrow	(2,5)	13
10	(1,6)	\longrightarrow	$(\gamma(1),6)$	\longrightarrow	(1,6)	10
11	(1,7)	\longrightarrow	$(\gamma(1),7)$	\longrightarrow	(1,7)	11
12	(2,4)	\longrightarrow	$(\gamma(2),4)$	\longrightarrow	(2,4)	12
13	(2,5)	\longrightarrow	$(\gamma(2),5)$	\longrightarrow	(1,5)	9
14	(2,6)	\longrightarrow	$(\gamma(2),6)$	\longrightarrow	(2,6)	14
15	(2,7)	\longrightarrow	$(\gamma(2),7)$	\longrightarrow	(2,7)	15
16	(3,4)	\longrightarrow	$(\gamma(3),4)$	\longrightarrow	(3,4)	16
17	(3,5)	\longrightarrow	$(\gamma(3),5)$	\longrightarrow	(3,5)	17
18	(3,6)	\longrightarrow	$(\gamma(3),6)$	\longrightarrow	(3,6)	18
19	(3,7)	\longrightarrow	$(\gamma(3),7)$	\longrightarrow	(3,7)	19

Table 9.8: Labeling of $\gamma_2(6)$

Labeling	Element				Element	Labeling
8	(1,4)	\longrightarrow	$(\gamma(1),4)$	\longrightarrow	(1,4)	8
9	(1,5)	\longrightarrow	$(\gamma(1),5)$	\longrightarrow	(1,5)	9
10	(1,6)	\longrightarrow	$(\gamma(1),6)$	\longrightarrow	(2,6)	14
11	(1,7)	\longrightarrow	$(\gamma(1),7)$	\longrightarrow	(1,7)	11
12	(2,4)	\longrightarrow	$(\gamma(2),4)$	\longrightarrow	(2,4)	12
13	(2,5)	\longrightarrow	$(\gamma(2),5)$	\longrightarrow	(2,5)	13
14	(2,6)	\longrightarrow	$(\gamma(2),6)$	\longrightarrow	(1,6)	10
15	(2,7)	\longrightarrow	$(\gamma(2),7)$	\longrightarrow	(2,7)	15
16	(3,4)	\longrightarrow	$(\gamma(3),4)$	\longrightarrow	(3,4)	16
17	(3,5)	\longrightarrow	$(\gamma(3),5)$	\longrightarrow	(3,5)	17
18	(3,6)	\longrightarrow	$(\gamma(3),6)$	\longrightarrow	(3,6)	18
19	(3,7)	\longrightarrow	$(\gamma(3),7)$	\longrightarrow	(3,7)	19

Table 9.9: Labeling of $\gamma_2(7)$

Labeling	Element				Element	Labeling
8	(1,4)	\longrightarrow	$(\gamma(1),4)$	\longrightarrow	(1,4)	8
9	(1,5)	\longrightarrow	$(\gamma(1),5)$	\longrightarrow	(1,5)	9
10	(1,6)	\longrightarrow	$(\gamma(1),6)$	\longrightarrow	(1,6)	10
11	(1,7)	\longrightarrow	$(\gamma(1),7)$	\longrightarrow	(2,7)	15
12	(2,4)	\longrightarrow	$(\gamma(2),4)$	\longrightarrow	(2,4)	12
13	(2,5)	\longrightarrow	$(\gamma(2),5)$	\longrightarrow	(2,5)	13
14	(2,6)	\longrightarrow	$(\gamma(2),6)$	\longrightarrow	(2,6)	14
15	(2,7)	\longrightarrow	$(\gamma(2),7)$	\longrightarrow	(1,7)	11
16	(3,4)	\longrightarrow	$(\gamma(3),4)$	\longrightarrow	(3,4)	16
17	(3,5)	\longrightarrow	$(\gamma(3),5)$	\longrightarrow	(3,5)	17
18	(3,6)	\longrightarrow	$(\gamma(3),6)$	\longrightarrow	(3,6)	18
19	(3,7)	\longrightarrow	$(\gamma(3),7)$	\longrightarrow	(3,7)	19

From the labeling we obtain $\gamma_2(4) = (8,12)$, $\gamma_2(5) = (9,13)$, $\gamma_2(6) = (10,14)$,
, and $\gamma_2(7) = (11,15)$.

Thus, from all the computations we have:

$$\gamma_1(4) = (8,12,16), \gamma_1(5) = (9,13,17), \gamma_1(6) = (10,14,18), \text{ and } \gamma_1(7) = (11,15,19).$$

$$\gamma_2(4) = (8,12), \gamma_2(5) = (9,13), \gamma_2(6) = (10,14), \text{ and } \gamma_2(7) = (11,15).$$

We have formed $B = H(4) \times H(5) \times H(6) \times H(7)$, and obtain four copies of S_3 . This can be expressed as $B = \langle \gamma_1(4), \gamma_2(4) \rangle \times \langle \gamma_1(5), \gamma_2(5) \rangle \times \langle \gamma_1(6), \gamma_2(6) \rangle \times \langle \gamma_1(7), \gamma_2(7) \rangle$. We label each as $a = (8,12,16)$, $b = (8,12)$, $c = (9,13,17)$, $d = (9,13)$, $e = (10,14,18)$, $f = (10,14)$, $g = (11,15,19)$, and $h = (11,15)$ such that $B = \langle a, b \rangle \times \langle c, d \rangle \times \langle e, f \rangle \times \langle g, h \rangle$.

We now compute k^* , where $K = \langle (4,5)(6,7), (4,6)(5,7) \rangle = 2^2$. Since it is not transitive we compute $k_1^* = (4,5)^*$ and $k_2^* = (6,7)^*$. We proceed as we did above.

Table 9.10: Labeling of $k^* = (4, 5)^*$

Labeling	Element		Element	Labeling
8	(1,4)	\longrightarrow	(1,5)	9
9	(1,5)	\longrightarrow	(1,4)	8
10	(1,6)	\longrightarrow	(1,6)	10
11	(1,7)	\longrightarrow	(1,7)	11
12	(2,4)	\longrightarrow	(2,5)	13
13	(2,5)	\longrightarrow	(2,4)	12
14	(2,6)	\longrightarrow	(2,6)	14
15	(2,7)	\longrightarrow	(2,7)	15
16	(3,4)	\longrightarrow	(3,5)	17
17	(3,5)	\longrightarrow	(3,4)	16
18	(3,6)	\longrightarrow	(3,6)	18
19	(3,7)	\longrightarrow	(3,7)	19

From this labeling we obtain $k_1^* = (8,9)(12,13)(16,17)$, which we label as $i = (8,9)(12,13)(16,17)$.

Table 9.11: Labeling of $k^* = (6, 7)^*$

Labeling	Element		Element	Labeling
8	(1,4)	\longrightarrow	(1,4)	8
9	(1,5)	\longrightarrow	(1,5)	9
10	(1,6)	\longrightarrow	(1,7)	11
11	(1,7)	\longrightarrow	(1,6)	10
12	(2,4)	\longrightarrow	(2,4)	12
13	(2,5)	\longrightarrow	(2,5)	13
14	(2,6)	\longrightarrow	(2,7)	15
15	(2,7)	\longrightarrow	(2,6)	14
16	(3,4)	\longrightarrow	(3,4)	16
17	(3,5)	\longrightarrow	(3,5)	17
18	(3,6)	\longrightarrow	(3,7)	19
19	(3,7)	\longrightarrow	(3,6)	18

We obtain $k_2^* = (11,10)(14,15)(18,19)$, which we label as $j = (11,10)(14,15)(18,19)$.

Therefore, $k^* = \langle k_1^*, k_2^* \rangle = \langle i, j \rangle$.

Thus, the wreath product $H \wr K$, such that $G = B : k^* = \langle a, b, c, d, e, f, g, h \rangle : \langle i, j \rangle$ is $S_3 \wr 2^2$.

9.1.2 Example: A Presentation of $S_3 \wr 2^2$

We now wish to write a presentation for the previous example. To do this we must compute the action of $K=2^2$ on $H=S_3$. Then since H is composed of 4 copies of S_3 which is generated by $\langle a, b, c, d, e, f, g, h \rangle$ and $2^2 = \langle i, j \rangle$. We compute

$$\begin{aligned} a^i &= (8, 12, 16)^{(8,9)(12,13)(16,17)} = (9,13,17) = c, \\ a^j &= (8, 12, 16)^{(11,10)(14,15)(18,19)} = (8,12,16) = a, \\ b^i &= d, b^j = b, \\ c^i &= a, c^j = c, \\ d^i &= b, d^j = d, \\ e^i &= e, e^j = g, \\ f^i &= f, f^j = h, \\ g^i &= g, g^j = e, \\ h^i &= h, h^j = f, \end{aligned}$$

Thus, a presentation for $S_3 \wr 2^2$ is given by:

$$\begin{aligned} &\langle a, b, c, d, e, f, g, h, i, j \mid a^3, b^2, (ab)^2, c^3, d^2, (cd)^2, e^3, f^2, (ef)^2, g^3, h^2, (gh)^2, \\ &(a,b), (a,c), (a,d), (a,e), (a,f), (a,g), (a,h), (b,c), (b,d), (b,e), (b,f), (b,g), (b,h), \\ &(c,d), (c,e), (c,f), (c,g), (c,h), (d,e), (d,f), (d,g), (d,h), (e,f), (e,g), (e,h), (f,g), (f,h), \end{aligned}$$

$$(g,h), i^2, j^2, a^i = c, a^j = a, b^i = d, b^j = b, c^i = a, c^j = c, d^i = b, d^j = d, e^i = e, e^j = g, f^i = f, f^j = h, g^i = g, g^j = e, h^i = h, h^j = f >.$$

9.1.3 Example: A Presentation of $S_4 \wr S_3$

In this example we wish to find the presentation for $S_4 \wr S_3$ without manually finding the permutation generators by hand.

We begin, by inspection we see that S_3 acts on three letters this implies our presentation will have three copies of S_4 . The three copies of S_4 are represented below.

$$a^4, b^2, (ab)^3$$

$$c^4, d^2, (cd)^3$$

$$e^4, f^2, (ef)^3$$

The elements of S_4 commute so additionally we will have (a,c), (a,d), (a,e), (a,f), (b,c), (b,d), (b,e), (b,f), (c,e), (c,f), (d,e), and (d,f).

We know S_3 can be generated by (1,2,3) and (1,2)(3) = (1,2). Let $g = (1,2,3)$ and $h = (1,2)(3) = (1,2)$ then we will label our three copies of S_4 by $S_4^3 = \langle a, b \rangle \times \langle c, d \rangle \times \langle e, f \rangle$ as :

$$\langle a, b \rangle = 1$$

$$\langle c, d \rangle = 2$$

$$\langle e, f \rangle = 3$$

We now compute the action of g and h on the three copies of S_4 .

$$a^g = a^{(1,2,3)} = 1^{(1,2,3)} = 2 \implies a^g = c$$

$$a^h = a^{(1,2)} = 1^{(1,2)} = 2 \implies a^h = c$$

$$b^g = b^{(1,2,3)} = 1^{(1,2,3)} = 2 \implies b^g = d$$

$$b^h = b^{(1,2)} = 1^{(1,2)} = 2 \implies b^h = d$$

$$c^g = c^{(1,2,3)} = 2^{(1,2,3)} = 3 \implies c^g = e$$

$$c^h = c^{(1,2)} = 2^{(1,2)} = 1 \implies c^h = a$$

$$d^g = d^{(1,2,3)} = 2^{(1,2,3)} = 3 \implies d^g = f$$

$$d^h = d^{(1,2)} = 2^{(1,2)} = 1 \implies d^h = b$$

$$e^g = e^{(1,2,3)} = 3^{(1,2,3)} = 1 \implies e^g = a$$

$$e^h = e^{(1,2)} = 3^{(1,2)} = 3 \implies e^h = e$$

$$f^g = f^{(1,2,3)} = 3^{(1,2,3)} = 1 \implies f^g = b$$

$$f^h = f^{(1,2)} = 3^{(1,2)} = 3 \implies f^h = f$$

Thus, a presentation for $S_4 \wr S_3$ is

$$\langle a, b, c, d, e, f, g, h \mid a^4, b^2, (ab)^3, c^4, d^2, (cd)^3, e^4, f^2, (ef)^3, (a,c), (a,d), (a,e), \\ (a,f), (b,c), (b,d), (b,e), (b,f), (c,e), (c,f), (d,e), (d,f), g^3, h^2, a^g = c, a^h = c, b^g = \\ d, b^h = d, c^g = e, c^h = a, d^g = f, d^h = b, e^g = a, e^h = e, f^g = b, f^h = f \rangle.$$

9.1.4 Example: Verifying Wreath Product $S_4 \wr S_3$

In order to be able to build a progenitor with MAGMA we need to make sure H and K are transitive. We can verify in MAGMA to see if the presentation for the wreath product is correct with the following loop:

```
P:=WreathProduct(Sym(4), Sym(3));
```

```

G<a,b,c,d,e,f,g,h>:=Group<a,b,c,d,e,f,g,h| a4, b2, (a * b)3, c4, d2, (c * d)3,
e4, f2, (e * f)3, g3, h2, (g * h)2, (a, c), (a, d), (a, e), (a, f), (b, c), (b, d), (b, e), (b, f),
(c, e), (c, f), (d, e), (d, f), ag = c, ah = c, bg = d, bh = d, cg = e, ch = a, dg = f,
dh = b, eg = a, eh = e, fg = b, fh = f >;
f, G1, k := CosetAction(G, sub < G | Id(G) >);
#P;
82944
#G1;
82944
IsIsomorphic(G1, P);
trueMappingfrom : GrpPerm : G1toGrpPerm : P
CompositionofMappingfrom : GrpPerm : G1toGrpPCand
Mappingfrom : GrpPCtoGrpPCand
Mappingfrom : GrpPCtoGrpPerm : P

```

Thus, the presentation is correct.

9.1.5 Example: Reducing the Number of Generators in the Presentation of $S_4 \wr S_3$

We wish to find the wreath product of S_4 by S_3 without finding the permutation generators as we did in the previous example.

$H = S_4$ and $K = S_3$, K is on 3 letters so we will have three copies of H .

The three copies of S_4 will be generated by the following:

$$a^4, b^2, (ab)^3$$

$$c^4, d^2, (cd)^3$$

$$e^4, f^2, (ef)^3$$

Similarly, S_3 is generated by : $i^3, j^2, (ij)^2$

We can now write the presentation for G :

$$\begin{aligned} G \langle a, b, c, d, e, f, i, j \rangle := & \text{Group} \langle a, b, c, d, e, f, i, j \mid a^4, b^2, (a * b)^3, c^4, d^2, (c * d)^3, e^4, \\ & f^2, (e * f)^3, i^3, j^2, (i * j)^2, (a, c), (a, d), (a, e), (a, f), (b, c), (b, d), (b, e), (b, f), (c, e), \\ & (c, f), (d, e), (d, f), a^i = c, b^i = d, c^i = e, d^i = f, e^i = a, f^i = b, a^j = c, b^j = d, c^j = \\ & a, d^j = b, e^j = e, f^j = f \rangle; \end{aligned}$$

We find that the order of G is $24^3 * 6 = 82,944$.

Using the following loop to find a minimal faithful permutation representation for $G1$:

```
SL:=Subgroups(G1);
T := {X'subgroup: X in SL};
#T;
TrivCore := {H:H in T | #Core(G1,H) eq 1};
mdeg := Min({Index(G1,H):H in TrivCore});
Good := {H: H in TrivCore | Index(G1,H) eq mdeg};
#Good;
H := Rep(Good);
```

#H;

f,G1,K := CosetAction(G1,H);

#W;

#G1;

we find the minimal faithful permutation representation for G1 has cardinality 12 having the following generators: a= (3, 5, 8, 10), b=(3, 5), c= (6, 9, 11, 12), d= (6, 9), e= (1, 2, 4, 7), f= (1, 2), i= (1, 3, 6)(2, 5, 9)(4, 8, 11)(7, 10, 12), j= (3, 6)(5, 9)(8, 11)(10, 12). However, we can reduce the number of generators for G by treating c= (6, 9, 11, 12), d= (6, 9), e= (1, 2, 4, 7), f= (1, 2) as redundant because $a^i = c$, $(a^i)^i = e$, $b^i = d$, $c^i = e$, and $(b^i)^i = f$. We check if this is true by doing the following:

sub < Sym(12)|(3, 5, 8, 10), (3, 5), (1, 3, 6)(2, 5, 9)(4, 8, 11)(7, 10, 12), (3, 6)(5, 9)(8, 11)(10, 12) > eq
sub < Sym(12)|(3, 5, 8, 10), (3, 5), (1, 3, 6)(2, 5, 9)(4, 8, 11)(7, 10, 12), (3, 6)(5, 9)(8, 11)(10, 12), (6, 9, 11, 12), (6, 9), (1, 2, 4, 7), (1, 2) >;

Now, write the relations form the original presentation in terms of these and the presentation now becomes :

$H < a, b, i, j > := Group < a, b, i, j | a^4, b^2, (a * b)^3, (a^i)^4, (b^i)^2, ((a^i) * (b^i))^3, ((a^i)^i)^4, ((b^i)^i)^2, (((a^i)^i) * ((b^i)^i))^3, i^3, j^2, (i * j)^2, (a, (a^i)), (a, (b^i)), (a, ((a^i)^i)), (a, ((b^i)^i)), (b, (a^i)), (b, (b^i)), (b, ((a^i)^i)), (b, ((b^i)^i)), ((a^i), ((a^i)^i)), ((a^i), ((b^i)^i)), ((b^i), ((a^i)^i)), ((b^i), ((b^i)^i)), a^i = (a^i), b^i = (b^i), (a^i)^i = ((a^i)^i), (b^i)^i = ((b^i)^i), ((a^i)^i)^i = a, ((b^i)^i)^i = b, a^j = (a^i), b^j = (b^i), (a^i)^j = a, (b^i)^j = b, ((a^i)^i)^j = ((a^i)^i), ((b^i)^i)^j = ((b^i)^i) >;$

We do a quick check in magma to see if we have successfully re-written the presentation with fewer generators:

```
G< a,b,c,d,e,f,i,j >:=Group< a,b,c,d,e,f,i,j|a^4,b^2,(a*b)^3,c^4,d^2,(c*d)^3,e^4,f^2,
(e*f)^3,i^3,j^2,(i*j)^2,(a,c),(a,d),(a,e),(a,f),(b,c),(b,d),(b,e),(b,f),(c,e),(c,f),
(d,e),(d,f),a^i=c,b^i=d,c^i=e,d^i=f,e^i=a,f^i=b,a^j=c,b^j=d,c^j=a,d^j=
b,e^j=e,f^j=f >;
```

```
H< a,b,i,j >:=Group< a,b,i,j|a^4,b^2,(a*b)^3,(a^i)^4,(b^i)^2,((a^i)*(b^i))^3,((a^i)^i)^4,
((b^i)^i)^2,(((a^i)^i)*((b^i)^i))^3,i^3,j^2,(i*j)^2,(a,(a^i)),(a,(b^i)),(a,((a^i)^i)),(a,((b^i)^i)),
(b,(a^i)),(b,(b^i)),(b,((a^i)^i)),(b,((b^i)^i)),((a^i),((a^i)^i)),((a^i),((b^i)^i)),((b^i),((a^i)^i)),
((b^i),((b^i)^i)),a^i=(a^i),b^i=(b^i),(a^i)^i=((a^i)^i),(b^i)^i=((b^i)^i),((a^i)^i)^i=a,((b^i)^i)^i=
b,a^j=(a^i),b^j=(b^i),(a^i)^j=a,(b^i)^j=b,((a^i)^i)^j=((a^i)^i),((b^i)^i)^j=((b^i)^i) >;
```

```
f,G1,k:=CosetAction(G,sub<G|Id(G)>);
```

```
f,H1,k:=CosetAction(H,sub<H|Id(H)>);
```

```
#G;
```

```
82944
```

```
#H; 82944
```

```
IsIsomorphic(G1,H1);
```

```
true Mapping from: GrpPerm: G1 to GrpPerm: H1 Composition of Mapping
from: GrpPerm: G1 to GrpPC and Mapping from: GrpPC to GrpPC and Map-
ping from: GrpPC to GrpPerm: H1
```

Thus, we are successful and we can proceed to build a progenitor if desired.

Appendix A

MAGMA CODE Progenitor

$$2^6 : (3^2 : 2^2)$$

```

N:=TransitiveGroup(6,9);
S:=Sym(6);
xx:=S!(2, 4, 6);
yy:=S!(1, 5)(2, 4);
zz:=S!(1, 4)(2, 5)(3, 6);
N:=sub<S|xx,yy,zz>;
#N;
FPGroup(N);

G<x,y,z>:=Group<x,y,z|x^3,y^2,z^2,(x^-1 * y)^2,(y * z)^2,x^-1 * z * x^-1
* z * x * z * x * z>;
#G;
NN<a,b,c>:=Group<a,b,c|a^3,b^2,c^2,(a^-1*b)^2,(b*c)^2,a^-1
*c*a^-1*c*a*c*a*c>;
#NN;
f,NN1,k:=CosetAction(NN,sub<NN|Id(NN)>);
IsIsomorphic(NN1,N);
N1:=Stabilizer(N,1);
Generators(N1);

```

```

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..#N]];
for i in [2..#N] do
P:=[Id(N): 1 in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=zz; end if;end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;end for;

for i in [1..#Sch] do if ArrayP[i] eq N!(2,4,6) then Sch[i];
end if; end for;
for i in [1..#Sch] do if ArrayP[i] eq N!(3,5)(4,6) then Sch[i];
end if; end for;

N1 eq sub<N|xx,xx*yy*zz*xx*zz>;
C:=Classes(N); C;
Orbits(N1);

G<x,y,z,t>:=Group<x,y,z,t| x^3,y^2,z^2,(x^-1 * y)^2,(y * z)^2,
x^-1 * z * x^-1 * z * x * z * x * z,t^2,(t,x*y*z*x*z),(t,x),
(t,t^(y*z)),(t,t^(y*z^2))>;
#G;

2^6*36;

for i in [2..#C] do i, C[i][3];
for j in [1..#N] do if ArrayP[j] eq C[i][3] then Sch[j]; end if; end for;
Orbits(Centralizer(N,C[i][3])); end for;
for j in [1..6] do for i in [1..#N] do if 1^ArrayP[i] eq j then Sch[i];
break;end if; end for; end for;

for a,b,c,d,e,f,g,h,i,j in [0..10] do
G<x,y,z,t>:=Group<x,y,z,t| x^3,y^2,z^2,(x^-1 * y)^2,

```



```

(y * z)^2, x^-1 * z * x^-1 * z * x * z * x * z, t^2, (t, x*y*z*x*z), (t, x),
(z*t)^a, (y*z*t)^b, (y*t)^c, (y*t^(z*x*z))^d, ((x*z)^2*t)^e,
(x*z*x^-1*z*t)^f, (z*x*z*t)^g, (z*x*z*t^(y*z))^h, (z*x*t)^i, (y*z*x*t)^j>;
if Index(G, sub<G|x,y,z>) gt 1 then a,b,c,d,e,f,g,h,i,j,
Index(G, sub<G|x,y,z>); end if; end for;

```

Appendix B

MAGMA CODE Monomial

Progenitor $17^4 :_m (2^3 : 4)$

```

S:=Sym(64);
xx:=S!(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)
(23,24)(25,26)(27,28)(29,30)(31,32)(33,34)(35,36)(37,38)(39,40)(41,42)
(43,44)(45,46)(47,48)(49,50)(51,52)(53,54)(55,56)(57,58)(59,60)(61,62)
(63,64);

yy:=S!(1,5,13,29,61,57,49,33)(2,3,50,51,62,63,14,15)(4,60,16,36,64,8,52,32)
(6,7,34,35,58,59,30,31)(9,21,45,25,53,41,17,37)(10,11,18,19,54,55,46,47)
(12,44,48,40,56,24,20,28)(22,23,38,39,42,43,26,27);
N:=sub<S|xx, yy>;
#N;

FPGGroup(N);

H<x,y>:=Group<x,y| x^2,(y * x * y)^2,y^8, (y * x * y^-1 * x)^2>;
#H;
NN<a,b>:=Group<a,b|a^2,(b * a * b)^2,b^8, (b * a * b^-1 * a)^2>;
#NN;
f,NN1,k:=CosetAction(NN,sub<NN|Id(NN)>);

```

```

IsIsomorphic(NN1,N);

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..#N]];
for i in [2..#N] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;end for;

Stabilizer(N,{1,5,9,13,17,21,25,29,33,37,41,45,49,53,57,61});
yy^3;

Orbits(N);
/* we find this through monomial presentation and we use this to
test progenitor
Stabilizer(G,1);
Orbits(Stabilizer(G,1)); */

1^(xx*yy);
1^(yy);
1^(xx);
1^(xx*yy);
1^(xx*yy*xx);
1^(yy^3);

C:=Classes(N);C;
H<x,y,t>:=Group<x,y,t| x^2,(y * x * y)^2,y^8, (y * x * y^-1 * x)^2,t^17,
t^(y^3)=t^8,(t,t^(x*y)),(t,t^y),(t,t^x),(t,t^(x*y*x))>;
#H;

(17)^4*32;
/* without (t,t^y) */

```

```

H<x,y,t>:=Group<x,y,t| x^2,(y * x * y)^2,y^8, (y * x * y^-1 * x)^2,t^17,
t^(y^3)=t^8,(t,t^(x*y)),(t,t^x),(t,t^(x*y*x))>;
#H;
(17)^4*32;

for i in [2..#C] do i, C[i][3];
for j in [1..#N] do if ArrayP[j] eq C[i][3] then Sch[j];
end if; end for;
Orbits(Centralizer(N,C[i][3])); end for;

A:={1, 2, 5, 3, 6, 13, 4, 50, 7, 14, 29, 60, 49, 51, 8, 34, 15, 30,
61,59, 16, 33, 52, 62, 35, 31, 57, 36, 32, 63, 58, 64 };

B:={9, 10, 21, 11, 22, 45, 12, 18, 23, 46, 25, 44, 17, 19, 24, 38, 47,
26, 53, 43, 48, 37, 20, 54, 39, 27, 41, 40, 28, 55, 42, 56};

for j in A do for i in [1..#N] do if 1^ArrayP[i] eq j then j,
Sch[i]; break; end if; end for; end for;

for j in B do for i in [1..#N] do if 1^ArrayP[i] eq j then j,
Sch[i]; break; end if; end for; end for;

for a,b,c,d,e,f,g,h,i,j in [0..10] do
G<x,y,t>:=Group<x,y,t| x^2,(y * x * y)^2,y^8,(y * x * y^-1 * x)^2,t^17,
t^(y^3)=t^8,(y^4*t)^a,((x*y*x*y^-1)*t)^b,(x*t)^c, ((x*y^2)*t)^d,(y*t)^e,
(y^3*t)^f,(y^2*t)^g,((x*y)^2*t)^h,((x*y)*t)^i,((y*x*y^-1)*t)^j>;
if Index(G,sub<G|x,y>) gt 1 then a,b,c,d,e,f,g,h,i,j,
Index(G,sub<G|x,y>); end if; end for;

```

Appendix C

MAGMA CODE $3^2 : 2^2$

```

N:=TransitiveGroup(6,9);
N;
S:=Sym(6);
xx:=S!(2,4,6);
yy:=S!(1,5)(2,4);
zz:=S!(1,4)(2,5)(3,6);
N:=sub<S|xx,yy,zz>;
N;
#N;
NL:=NormalLattice(N);
NL;
for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if; end for;
NL[6];
A:=N!(2,4,6);
B:=N!(1,3,5)(2,6,4);
IsIsomorphic(NL[6],AbelianGroup(GrpPerm,[3,3]));
/* NL[6] is isomorphic to 3x3 */
G<a,b>:=Group<a,b|a^3,b^3,(a,b)>;
#G;
f,G1,k:=CosetAction(G,sub<G|Id(G)>);

IsIsomorphic(G1,NL[6]);
q,ff:=quo<N|NL[6]>;

```

```

q;
FPGGroup(q);
CompositionFactors(q);
nl:=NormalLattice(q);
nl;

for i in [1..#nl] do if IsAbelian(nl[i]) then i; end if; end for;
nl[5];
/* presentation for q*/
H<c,d,e>:=Group<c,d,e|d^2,e^2,(d*e)^2,c>;
#H;
IsIsomorphic(q,AbelianGroup(GrpPerm,[2,2]));
/* q is isomorphic to 2*2=2^2*/
H<c,d,e>:=Group<c,d,e|d^2,e^2,(d*e)^2,c>;
#H;

f,H1,k:=CosetAction(H,sub<H|Id(H)>);
IsIsomorphic(H1,q);

T:=Transversal(N,NL[6]);
T;
ff(T[1]) eq q.1;
ff(T[2]) eq q.2;
ff(T[3]) eq q.3;

A;B;A*B;A^2*B;A*B^2;(A*B)^2;A^2,B^2;
A^T[1]eq A;A^T[2] eq A^2; A^T[3] eq A*B;
B^T[1] eq B; B^T[2] eq B^2; B^T[3] eq B^2;
J<a,b,c,d,e>:=Group<a,b,c,d,e|a^3,b^3,(a,b),d^2,e^2,(d*e)^2,
c,a^c=a,a^d=a^2,a^e=a*b,b^c=b,b^d=b^2,b^e=b^2>;
#J;
f,J1,k:=CosetAction(J,sub<J|Id(J)>);
IsIsomorphic(J1,N);
/* so isomorphism type of N=T(6,9) is 3^2:2^2*/

```

Appendix D

MAGMA CODE Mixed

Extension $2^7 : \bullet 2^6$

```

N:=TransitiveGroup(28,590);
#N;
N;

S:=Sym(28);
xx:=S!(1, 6, 12, 16, 20, 23, 27, 4, 8, 9, 13, 18, 22, 25)
(2, 5, 11, 15, 19,24,28, 3, 7, 10, 14, 17, 21, 26);
yy:=S!(1, 12, 17, 25, 6, 15, 23, 3, 10, 20, 28, 8, 14, 22)
(2, 11, 18, 26, 5,16,24, 4, 9, 19, 27, 7, 13, 21);
NL:=NormalLattice(N);
NL;
CompositionFactors(N);

for i in [1..#NL] do if IsAbelian(NL[i]) then i;
end if;end for;
NL[5];
A:=N!(1, 3)(2, 4)(5, 8)(6, 7)(9, 12)(10, 11)(13, 15)(14, 16)(17, 20)(18, 19)
(21,23)(22, 24)(25, 28)(26, 27) ;
B:=N!(21, 22)(23, 24)(25, 26)(27, 28);
C:=N!(17, 18)(19, 20)(21, 22)(23, 24);

```

```

D:=N!(13, 14)(15, 16)(25, 26)(27, 28);
E:=N!(1, 2)(3, 4)(5, 6)(7, 8)(17, 18)(19, 20)(25, 26)(27, 28);
F:=N!(5, 6)(7, 8)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24);
G:=N!(9, 10)(11, 12)(17, 18)(19, 20)(21, 22)(23, 24)(25, 26)(27, 28);
#NL[5];

IsIsomorphic(NL[5],AbelianGroup(GrpPerm,[2,2,2,2,2,2,2]));

H<a,b,c,d,e,f,g>:=Group<a,b,c,d,e,f,g|a^2,b^2,c^d^2,e^2,f^2,g^2,
(a,b),(a,c),(a,d),(a,e),(a,f),(a,g)(b,c),(b,d),(b,e),(b,f),(b,g),
(c,d),(c,e),(c,f),(c,g),(e,f),(e,g)>;

q,ff:=quo<N|NL[5]>; q;
#q;
nl:=NormalLattice(q); nl;

for i in [1..#nl] do if IsAbelian(nl[i]) then i;end if;end for;
nl[4];

FPGGroup(q);
/*<h,j>:=<h,j|h^7,j^7,(h * j^-1 * h)^2,(h^-1, j^-1)^2,(h^-1 * j^-3)^2>;*/

IsIsomorphic(nl[4],AbelianGroup(GrpPerm,[2,2,2,2,2,2,2]));

J<h,j>:=Group<h,j|h^7,j^7,(h * j^-1 * h)^2,(h^-1, j^-1)^2,(h^-1 * j^-3)^2>;
f,J1,k:=CosetAction(J,sub<J|Id(J)>);
IsIsomorphic(J1,q);

T:=Transversal(N,NL[5]);
ff(T[2]) eq q.1;
ff(T[3]) eq q.2;

A^T[2];
A^T[3];
B^T[2];
B^T[3];
C^T[2];
C^T[3];

```



```

D^T[2];
D^T[3];
E^T[2];
E^T[3];
F^T[2];
F^T[3];
G^T[2];
G^T[3];
A^T[2] eq A*C*D*G;
A^T[3] eq A*C*E*G;
B^T[2] eq B*D*E*F;
B^T[3] eq B*C*E;
C^T[2] eq B;
C^T[3] eq B*D*E*F;
D^T[2] eq C*D*E*F;
D^T[3] eq B*C*D*F;
E^T[2] eq E*G;
E^T[3] eq F*G;
F^T[2] eq G;
F^T[3] eq E*F;
G^T[2] eq E*F;
G^T[3] eq E;

M<a,b,c,d,e,f,g,h,j>:=Group<a,b,c,d,e,f,g,h,j|a^2,b^2,c^2,d^2,e^2,f^2,g^2,
(a,b),(a,c),(a,d),(a,e),(a,f),(a,g),(b,c),(b,d),(b,e),(b,f),(b,g),
(c,d),(c,e),(c,f),(c,g),(e,f),(e,g),h^7,j^7,(h * j^-1 * h)^2,
(h^-1, j^-1)^2,(h^-1 * j^-3)^2>;
f,M1,k:=CosetAction(M,sub<M|Id(M)>);
IsIsomorphic(N,M1);

#M;
#N;
/* SO far we seen ISO TYPE is semi direct product 2^7 : 2^6
but now since we got false we need to find elements of N/NL[5]
in terms of NL[5]*/

/* WE MUST CHECK :
h^7,
j^7,
```

```

(h * j^-1 * h)^2,
(h^-1, j^-1)^2,
(h^-1 * j^-3)^2 */

Order(q.1);
Order(T[2]);
Order(q.2); Order(T[3]);
Order(q.2); Order(T[3]^2);
Order(q.1); Order(T[2]^2);

(q.1 * q.2^-1 * q.1)^2; (T[2] * T[3]^-1 * T[2])^2;
/* SO THIS ONE IS GOOD, IT DOESNT CHANGE*/

(q.1^-1 , q.2^-1)^2; (T[2]^-1 , T[3]^-1)^2;
/*THIS ONE IS GOOD TOO SO WE DONT CHNAGE IT*/

(q.1^-1 * q.2^-3)^2; (T[2]^-1 * T[3]^-3)^2;
/* WE NEED TO CHANGE THIS ONE*/

T[2]^7;
T[3]^7;

for i,j,k,l,m,n,o in [0..1] do if T[2]^7 eq A^i*B^j*C^k*D^l*E^m*F^n*G^o
then i,j,k,l,m,n,o; end if; end for;

for i,j,k,l,m,n,o in [0..1] do if T[3]^7 eq A^i*B^j*C^k*D^l*E^m*F^n*G^o
then i,j,k,l,m,n,o; end if; end for;

for i,j,k,l,m,n,o in [0..1] do if (T[2]^-1 * T[3]^-3)^2
eq A^i*B^j*C^k*D^l*E^m*F^n*G^o
then i,j,k,l,m,n,o; end if; end for;

T[2]^7 eq A*B*D*E;

T[3]^7 eq A*B*F*G;
(T[2] * T[3]^-1 * T[2])^2;
(T[2]^-1, T[3]^-1)^2;
(T[2]^-1 * T[3]^-3)^2 eq B*F;

```

```

M<a,b,c,d,e,f,g,h,j>:=Group<a,b,c,d,e,f,g,h,j|a^,b^2,c^2,d^2,e^2,f^2,
g^2,(a,b),(a,c),(a,d),(a,e),(a,f),(a,g),(b,c),(b,d),(b,e),(b,f),(b,g),
(c,d),(c,e),(c,f),(c,g),(e,f),(e,g),h^7=a*b*d*e, j^7=a*b*f*g,
(h * j^-1 * h)^2,(h^-1, j^-1)^2,(h^-1 * j^-3)^2=b*f,
a^h=a*c*d*g , a^j=a*c*e*g , b^h=b*d*e*f , b^j= b*c*e, c^h=b ,
c^j=b*d*e*f, d^h=c*d*e*f , d^j=b*c*d*f ,
e^h=e*g, e^j=f*g, f^h=g, f^j=e*f, g^h=e*f, g^j=e>;

f,M1,k:=CosetAction(M,sub<M|Id(M)>);
IsIsomorphic(M1,N);

/* Hence Iso type is mixed extn 2^7:*2^6 */

```

Appendix E

MAGMA CODE DCE $3^3 : 2^4$ over $3^2 : 2^2$

```

N:=TransitiveGroup(6,9);
N;
S:=Sym(6);
xx:=S!(2,4,6);
yy:=S!(1,5)(2,4);
zz:=S!(1,4)(2,5)(3,6);
N:=sub<S|xx,yy,zz>;
G<x,y,z,t>:=Group<x,y,z,t| x^3,y^2,z^2,(x^-1 * y)^2,
(y * z)^2, x^-1 * z * x^-1 * z * x * z * x * z,t^2,
(t,x*y*z*x*z),(t,x),(z*x*t)^4, (t*t^(z*y))^2
/*(x*y*z*x*z*t*t^y)^2 redundant */
/* (y*z*t^x*t^(z*y))^2 redundant */>;

f, G1, k := CosetAction(G, sub< G | x, y,z>);
CompositionFactors(G1);
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);

IN:=sub<G1|f(x),f(y),f(z)>;
ts := [ Id(G1): i in [1 .. 6] ];
ts[1]:=f(t);

```

```

ts[2]:=f(t^(z*y));
ts[3]:=f(t^(z*x*z));
ts[4]:=f(t^z);
ts[5]:=f(t^y);
ts[6]:=f(t^(z*x));

1^((zz*xx)^3),
1^((zz*xx)^2),
1^(zz*xx);
f((x*z)^4)*ts[2]*ts[3] eq ts[6]*ts[1];

cst := [null : i in [1 .. 12]] where null is [Integers() | ];
prodim := function(pt, Q, I)
/*
Return the image of pt under permutations Q[I] applied sequentially.
*/
v := pt;
for i in I do
v := v^(Q[i]);
end for;
return v;
end function;
for i := 1 to 6 do
cst[prodim(1, ts, [i])] := [i];
end for;
m:=0; for i in [1..12] do if cst[i] ne [] then m:=m+1; end if; end for;m;

N1:=Stabiliser(N,1);
Orbits(N1);

N12:=Stabiliser(N1,2);
S:={[1,2]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[1]*ts[2]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;

```

```

N12s:=N12;
for g in N do if [1,2]^g eq [2,1] then N12s:=sub<N|N12s,g>; end if; end for;
for g in N do if [1,2]^g eq [3,4] then N12s:=sub<N|N12s,g>; end if; end for;
[1,2]^N12s;
#N12s;
Orbits(N12s);

N13:=Stabiliser(N1,3); N13;
N13s:=N13;
S:={[1,3]};
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[1]*ts[3]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;

for g in N do if [1,3]^g eq [6,4] then N13s:=sub<N|N13s,g>; end if; end for;
/* redundant
for g in N do if [1,3]^g eq [2,6] then N13s:=sub<N|N13s,g>; end if; end for;*/
#N13s, N13s,
Orbits(N13s);
[1,3]^N13s;

```

Appendix F

MAGMA CODE Proof $3^3 : 2^4$

```

T:=Sym(6);
xx:=T!(2,4,6);
yy:=T!(1,5)(2,4);
zz:=T!(1,4)(2,5)(3,6);

G<x,y,z,t>:=Group<x,y,z,t| x^3,y^2,z^2,(x^-1 * y)^2,
(y * z)^2, x^-1 * z * x^-1 * z * x * z * x * z,t^2,
(t,x*y*z*x*z),(t,x),(z*x*t)^4, (t*t^(z*y))^2
/*(x*y*z*x*z*t*t^y)^2 redundant */
/* (y*z*t^x*t^(z*y))^2 redundant */>;
/* last two realtions are redundant*/
f, G1, k := CosetAction(G, sub< G | x, y,z>);
N:=sub<T|xx,yy,zz>;
#N;
CompositionFactors(G1);
#G1;

/*ISOMORHISM TYPE OF G1 BEGINING*/

G<x,y,z,t>:=Group<x,y,z,t| x^3,y^2,z^2,(x^-1 * y)^2,
(y * z)^2, x^-1 * z * x^-1 * z * x * z * x * z,t^2,

```

```

(t,x*y*z*x*z),(t,x),(z*x*t)^4, (t*t^(z*y))^2
/*(x*y*z*x*z*t*t^y)^2 redundant */
/* (y*z*t^x*t^(z*y))^2 redundant */>;
/* last two realtions are redundant*/
f, G1, k := CosetAction(G, sub< G | x, y,z>);
#G;
CompositionFactors(G1);
NL:=NormalLattice(G1); NL;
for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if; end for;
NL[6];

A:=G1!(4, 5, 7)(8, 9, 12);
B:=G1!(1, 6, 10)(2, 3, 11)(4, 5, 7)(8, 9, 12);
C:=G1!(2, 11, 3)(4, 7, 5);
A*B;
A*C;
B*C;
FPGGroup(NL[6]);

H<a,b,c>:=Group<a,b,c| a^3, b^3, c^3, (a, b), (a, c), (b, c)>;#H;
f,H1,k:=CosetAction(H,sub<H|Id(H)>);
IsIsomorphic(H1,NL[6]);
q,ff:=quo<G1|NL[6]>; q;
FPGGroup(q);
CompositionFactors(q);
nl:=NormalLattice(q);nl;
for i in [1..#nl] do if IsAbelian(nl[i]) then i; end if; end for;

nl[16];

J<d,e,g>:=Group<d,e,g| d^2, e^2, g^2, (d * e)^2, (d * g)^2, (e * g)^4>; #J;
f,J1,k:=CosetAction(J,sub<J|Id(J)>);
IsIsomorphic(J1,nl[16]);
IsIsomorphic(J1,q);
T:=Transversal(G1,NL[6]); T;
ff(T[1]) eq q.1;
ff(T[2]) eq q.2;
ff(T[3]) eq q.3;
ff(T[4]) eq q.4;

```



```

M<a,b,c,d,e,g>:=Group<a,b,c,d,e,g|a^3, b^3, c^3,
(a, b), (a, c), (b, c),d^2, e^2, g^2,
(d * e)^2, (d * g)^2, (e * g)^4,
a^d=a^2,b^d=b^2,c^d=c^2, a^e=(a*c)^2,
b^e=b^2*c,c^e=c, a^g=a,b^g=b,c^g=(b*c)^2>;

```

```

f,M1,k:=CosetAction(M,sub<M|Id(M)>);
IsIsomorphic(M1,G1);

```

```

/* Isomorphism type of G1 is 3^3:2^4*/

```

```

/*END OF ISOMORPHISM TYPE OF G1*/

```

```

S:=Sym(12);
xxx:=S!(2,4,6);
yyy:=S!(1,5)(2,4);
zzz:=S!(1,4)(2,5)(3,6);

fx:=S!(3,5,7)(8,9,10);
Order(fx);
fy:=S!(2,6)(3,5)(8,10)(11,12);
Order(fy);
xx*yy;
Order(xx*yy);
Order(fx*fy);
fz:=S!(2,5)(3,6)(4,7)(8,10)(11,12); Order(fz);
#sub<S|fx,fy,fz>;
ft:=S!(1,2)(3,8)(4,12)(5,9)(6,11)(7,10);
Order(ft);
Order(xxx*yyy*zzz);
Order(fx*fy*fz);

#sub<S|fx,fy,fz,ft>;
G2:=sub<S|fx,fy,fz,ft>;
Order(G2);

```

```

IsIsomorphic(G1,G2);

```

```
s,t:=IsIsomorphic(M1,G2);
```

```
/*HENCE G2 is ISOMORPHIC TO  $3^3:2^4$ */
```

```
t(f(a));
```

```
t(f(b));
```

```
t(f(c));
```

```
t(f(d));
```

```
t(f(e));
```

```
t(f(g));
```

```
/* my isomorphism type of  $N=T(6,9)$  is  $3^2:2^2$  #G1 for DCE over T69 is 432  
so we have the same presentation for G2 but with these elements.  
now show they have the same order so then the kerf is 1.*/
```

Appendix G

MAGMA CODE DCE L(2,11)

over A_5

```
G<x,y,t>:=Group<x,y,t| x^5 , y^3 , (x^-1 * y^-1)^2 ,
x^-1 * y^-3 * x * y^-1 * x * y^3 * x^-1 * y ,
t^2, (t, x * y^2 * x^-2 * y),(x * t)^3>;
#G;
f,G1,k:=CosetAction(G,sub<G|x,y>);

NN<a,b>:=Group<a,b| a^5 , b^3 , (a^-1 * b^-1)^2 ,
a^-1 * b^-3 * a * b^-1 * a * b^3 * a^-1 * b>;
H:=sub<NN|a * b^2 * a^-2 * b>;
#NN;
#H;
f1,N,k:=CosetAction(NN,H);
N;

S:=Sym(20);
xx:=S!(1, 2, 6, 12, 4)(3, 9, 15, 16, 10)(5, 13, 8, 14, 7)
(11, 17, 19, 20, 18);
yy:=S!(1, 3, 5)(2, 7, 8)(4, 11, 9)(6, 13, 10)(12, 16, 17)
(15, 18, 19);
```

```

N:=sub<S|xx,yy>;
/* this will be 2^20:N*/
Stabiliser(N,1) eq sub<N|f1(a * b^2 * a^-2 * b)>;
#DoubleCosets(G,sub<G|x,y>, sub<G|x,y>);
DoubleCosets(G,sub<G|x,y>, sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
DC:=[Id(G1), f(t)];
ts := [ Id(G1): i in [1 .. 20] ];
ts[1]:=f(t);
ts[2]:=f(t^(x));
ts[3]:=f(t^(y));
ts[4]:=f(t^(x^4));
ts[5]:=f(t^(y^2));
ts[6]:=f(t^(x^2));
ts[7]:=f(t^(x*y));
ts[8]:=f(t^(x*(y^2)));
ts[9]:=f(t^(y*x));
ts[10]:=f(t^((x^2)*(y^2)));
ts[11]:=f(t^((x^-1)*y));
ts[12]:=f(t^(x^3));
ts[13]:=f(t^((x^2)*y));
ts[14]:=f(t^(y^2*x^3));
ts[15]:=f(t^(y*(x^2)));
ts[16]:=f(t^((x^3)*y));
ts[17]:=f(t^((x^3)*(y^2)));
ts[18]:=f(t^((x^-1)*y*(x^-1)));
ts[19]:=f(t^((x^-2)*(y^2)*x));
ts[20]:=f(t^((x^-1)*y*x^3));

Index(G,sub<G|x,y>);
cst := [null : i in [1 .. 11]] where null is [Integers() | ];
prodim := function(pt, Q, I)
v := pt;
for i in I do
v := v^(Q[i]);
end for;
return v;
end function;
for i := 1 to 20 do

```

```

cst[prodim(1, ts, [i])] := [i];
end for;
m:=0; for i in [1..11] do if cst[i] ne [] then m:=m+1; end if; end for;m;

Orbits(N);
N1:=Stabiliser(N,1);
#N1;
N1;
S:={[1]};S;
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]
eq g*ts[Rep(SSS[i])[1]]
then print SSS[i];
end if; end for; end for;
N1s:=sub<N|N1>;
tr1:=Transversal(N,N1s);
for i:=1 to #tr1 do
ss:=[1]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..11] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
for n in N do if 1^n eq 1 then n;N1s:=sub<N|N1s,n>;end if;end for;
for n in N do if 1^n eq 9 then n;N1s:=sub<N|N1s,n>;end if;end for;
N1s;
#N1s;/*PROVE THIS IF ITS NOT ALREADY A RELATION*/
#N/#N1s;
Orbits(N1s);

for i in [1..2] do
for g,h in IN do
if ts[1]*ts[1] eq g*(DC[i])^h then i; break; break;
end if; end for; end for;

for i in [1..2] do
for g,h in IN do

```

```

if ts[1]*ts[2] eq g*(DC[i])^h then i; break; break;
end if; end for; end for;

```

```

for i in [1..2] do
  for g,h in IN do
    if ts[1]*ts[4] eq g*(DC[i])^h then i; break; break;
    end if; end for; end for;

```

```

for i in [1..2] do
  for g,h in IN do
    if ts[1]*ts[6] eq g*(DC[i])^h then i; break; break;
    end if; end for; end for;

```

Appendix H

MAGMA CODE DCE U(3,5):2 over PGL(2,9)

```

G< x,y,t>:=Group<x,y,t|x^3,(x * y^-1)^2,y^10,
(x^-1 * y^-2)^4,y^3 * x^-1 * y^-4 * x * y^3 * x * y^-4 * x^-1 * y^2,
t^2,(t,x),(t,y^2 * x * y^-2), (y*t^(y^-2*x^-1))^10,
((y^3)*t^(y^-2*x^-1))^5, ((y^3)*t)^8 >;

/* THIS IS THE NEW ONE WITH THE NEW RELATION
we added in order to make relation proving easier*/

G< x,y,t>:=Group<x,y,t|x^3,(x * y^-1)^2,y^10,(x^-1 * y^-2)^4,
y^3 * x^-1 * y^-4 * x * y^3 * x * y^-4 * x^-1 * y^2,
t^2,(t,x),(t,y^2 * x * y^-2),
t*t^(((y^4)*x))=x*y*x*y^2*x*y^3*x*y^4*x*y^2*t*t^(y^8),
((y^3)*t^(y^-2*x^-1))^5, ((y^3)*t)^8 >;
f,G1,k:=CosetAction(G,sub<G|x,y>);
#G1;
CompositionFactors(G1);

T:=TransitiveGroups(12);
T[182];

```

```

S:=Sym(12); xx:=S!(2, 4, 10)(3, 5, 7)(6, 12, 8);
yy:=S!(1, 10, 9, 2, 7, 12, 11, 8, 5, 4)(3, 6);
N:=sub<S|xx,yy>;N;
#N;

DC:=[Id(G1), f( t), f( t * y * t), f( t * y^2* t),
f( t * y^2 * t * y^-2 * t), f( t * y * t * y* t),
f( t * y^2 * t * y * t)];
IN:=sub<G1|f(x),f(y)>;

ts:=[Id(G1): i in [1..12]];
ts[1]:=f(t);
ts[2]:=f(t^(y^3));
ts[3]:=f(t^((y^4)*x));
ts[4]:=f(t^(y^-1));
ts[5]:=f(t^(y^-2));
ts[6]:=f(t^((y^7)*x));
ts[7]:=f(t^(y^4));
ts[8]:=f(t^(y^7));
ts[9]:=f(t^(y^2));
ts[10]:=f(t^y);
ts[11]:=f(t^(y^6));
ts[12]:=f(t^(y^5));

N1:=Stabilizer(N,1);
N1;
Orbits(N1);

for i in [1..7] do/* the number of DC*/
for g,h in IN do
if ts[1]*ts[2] eq g*(DC[i])^h then i;
break; break; end if; end for;end for;

for i in [1..7] do
for g,h in IN do
if ts[1]*ts[3] eq g*(DC[i])^h then i;
break; break; end if; end for;end for;

cst := [null : i in [1 .. 350]] where null is [Integers() | ];

```



```

prodim := function(pt, Q, I)
v := pt;
for i in I do
v := v^(Q[i]);
end for;
return v;
end function;

for i:=1 to 12 do
cst[prodim(1, ts, [i])] := [i];
end for;
m:=0;for i in [1..350] do if cst[i] ne [] then m:=m+1; end if; end for;m;

N12:=Stabilizer(N,[1,2]);
N12;
S:={[1,2]};S;
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;

N12s:=sub<N|N12>;
tr12:=Transversal(N,N12s);
for i:=1 to #tr12 do
ss:=[1,2]^tr12[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..350] do if cst[i] ne []
then m:=m+1;
end if; end for;m;

for n in N do if 1^n eq 1 and 2^n eq 2 then n;
N12s:=sub<N|N12s,n>;end if;end for;
N12s;
#N12s;

```

```

720/10;
/* there are |N|/|N^(12)|=72*/
Orbits(N12s);

for i in [1..7] do
for g,h in IN do
if ts[1]*ts[2]*ts[1] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

for i in [1..7] do
for g,h in IN do
if ts[1]*ts[2]*ts[3] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

for i in [1..7] do
for g,h in IN do
if ts[1]*ts[2]*ts[4] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

for i in [1..7] do
for g,h in IN do
if ts[1]*ts[2]*ts[2] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

/* it is in DC f(t) =[1]*/

N13:=Stabilizer(N,[1,3]);
N13;

S:={[1,3]};S;

SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[3] /* we want to find which permutation g
in N are t1t3*/
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];

```

```

end if; end for; end for;

N13s:=sub<N|N13>;

tr13:=Transversal(N,N13s);
for i:=1 to #tr13 do
ss:=[1,3]^tr13[i];

cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..22] do if cst[i] ne []
then m:=m+1;
end if; end for;m;

for n in N do if 1^n eq 1 and 3^n eq 5 then
N13s:=sub<N|N13s,n>;
end if; end for;

for n in N do if 1^n eq 1 and 3^n eq 7 then
N13s:=sub<N|N13s,n>;
end if; end for;

for n in N do if 1^n eq 1 and 3^n eq 11 then
N13s:=sub<N|N13s,n>;
end if; end for;

for n in N do if 1^n eq 1 and 3^n eq 11 then
N13s:=sub<N|N13s,n>;
end if; end for;

N13s;
#N13s;
720/60;

/*|N|/|N^(13)|=12*/
Orbits (N13s);

for i in [1..7] do
for g,h in IN do

```

```

if ts[1]*ts[3]*ts[1] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

for i in [1..7] do
for g,h in IN do
if ts[1]*ts[3]*ts[3] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

for i in [1..7] do
for g,h in IN do
if ts[1]*ts[3]*ts[2] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

N123:=Stabilizer(N,[1,2,3]);
N123;
S:={[1,2,3]};
SS:=S^N; /* insert SS;*/
SSS:=Setseq(SS);
for i in [1..#SS] do for g in IN do if ts[1]*ts[2]*ts[3] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for;end for;

N123s:=N123;
for n in N do if 1^n eq 2 and 2^n eq 1 and 3^n eq 8 then
N123s:=sub<N|N123s,n>;
end if; end for;
N123s;
#N123s;

720/4;
/* |N|/|N^(123)|=180*/
[1,2,3]^N123s;

tr123:=Transversal(N,N123s);
for i:=1 to #tr123 do

```

```

ss:=[1,2,3]^tr123[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..350] do if cst[i] ne []
then m:=m+1;
end if; end for; m;

Orbits(N123s);

for i in [1..7] do
for g,h in IN do
if ts[1]*ts[2]*ts[3]*ts[1] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

for i in [1..7] do
for g,h in IN do
if ts[1]*ts[2]*ts[3]*ts[3] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

for i in [1..7] do
for g,h in IN do
if ts[1]*ts[2]*ts[3]*ts[4] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

for i in [1..7] do
for g,h in IN do
if ts[1]*ts[2]*ts[3]*ts[6] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

N124:=Stabilizer(N,[1,2,4]);
N124;
S:={[1,2,4]};
SS:=S^N; /* insert SS;*/

SSS:=Setseq(SS);
for i in [1..#SS] do for g in IN do if ts[1]*ts[2]*ts[4] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for;end for;

```

```

N124s:=N124;
for n in N do if 1^n eq 1 and 2^n eq 2 and 4^n eq 4 then
N124s:=sub<N|N124s,n>;end if; end for;

for n in N do if 1^n eq 11 and 2^n eq 2 and 4^n eq 10 then
N124s:=sub<N|N124s,n>;end if; end for;

for n in N do if 1^n eq 9 and 2^n eq 2 and 4^n eq 6 then
N124s:=sub<N|N124s,n>;end if; end for;

for n in N do if 1^n eq 3 and 2^n eq 2 and 4^n eq 12 then
N124s:=sub<N|N124s,n>;end if; end for;

for n in N do if 1^n eq 5 and 2^n eq 2 and 4^n eq 8 then
N124s:=sub<N|N124s,n>;end if; end for;

N124s;
#N124s;

[1,2,4]^N124s;

tr124:=Transversal(N,N124s);
for i:=1 to #tr124 do
ss:=[1,2,4]^tr124[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..350] do if cst[i] ne []
then m:=m+1;
end if; end for; m;

Orbits(N124s);

for i in [1..7] do
for g,h in IN do

```

```

if ts[1]*ts[2]*ts[4]*ts[2] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

for i in [1..7] do
  for g,h in IN do
    if ts[1]*ts[2]*ts[4]*ts[7] eq g*(DC[i])^h then i;
    break; break; end if; end for; end for;

for i in [1..7] do
  for g,h in IN do
    if ts[1]*ts[2]*ts[4]*ts[1] eq g*(DC[i])^h then i;
    break; break; end if; end for; end for;

for i in [1..7] do
  for g,h in IN do
    if ts[1]*ts[2]*ts[4]*ts[4] eq g*(DC[i])^h then i;
    break; break; end if; end for; end for;

N131:=Stabilizer(N,[1,3,1]);
N131;
S:={[1,3,1]};
SS:=S^N; /* insert SS;*/

SSS:=Setseq(SS);
for i in [1..#SS] do for g in IN do if
ts[1]*ts[3]*ts[1] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];end if; end for;end for;

N131s:=N131;
for n in N do if 1^n eq 1 and 3^n eq 3 and 1^n eq 1 then
N131s:=sub<N|N131s,n>;end if; end for;

for n in N do if 1^n eq 1 and 3^n eq 5 and 1^n eq 1 then
N131s:=sub<N|N131s,n>;end if; end for;

for n in N do if 1^n eq 1 and 3^n eq 7 and 1^n eq 1 then

```

```

N131s:=sub<N|N131s,n>;end if; end for;

for n in N do if 1^n eq 10 and 3^n eq 6 and 1^n eq 10 then
N131s:=sub<N|N131s,n>;end if; end for;

N131s;
#N131s;
[1,3,1]^N131s;

tr131:=Transversal(N,N131s);
for i:=1 to #tr131 do
ss:=[1,3,1]^tr131[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..350] do if cst[i] ne []
then m:=m+1;
end if; end for; m;

Orbits(N131s);

```


Appendix I

MAGMA CODE DCE $U(3,5):2$ over Maximal Subgroup $Aut(6)$ and $PGL(2,9)$

```

S:=Sym(12);
xx:=S!(2, 4, 10)(3, 5, 7)(6, 12, 8);
yy:=S!(1, 10, 9, 2, 7, 12, 11, 8, 5, 4)(3, 6);
N:=sub<S|xx,yy>;N;

a:=0; b:=0; c:=0; d:=0; e:=0; f:=0; g:=0; h:=0; i:=0;
j:=0; k:=0; l:=0; m:=0; n:=0; o:=0; p:=0; q:=0;
r:=0; s:=5; u:=8; v:=10; w:=8;

G< x,y,t>:=Group<x,y,t|x^3,(x * y^-1)^2,y^10,(x^-1 * y^-2)^4,
y^3 * x^-1 * y^-4 * x * y^3 * x * y^-4 * x^-1 * y^2,t^2,
(t,x),(t,y^2 * x * y^-2), ((y^5)*t^(y^-2*x^-1))^a,((y^5)*t)^b,
(((y*x*y)^2)*t^(y^-2*x^-1))^c,(((y*x*y)^2)*t)^d,(x*t)^e,
(x*t^(y*x))^f,(x*t^(y^-2*x^-1))^g,(x*t^(y^-2 *x^-1 * y))^h,
((y*x*y)*t^(y^-2*x^-1))^i ,((y*x*y)*t)^j,((y^2)*t^(y^-2*x^-1))^k,
((y^2)*t)^l ,((y^-4)*t^(y^-2*x^-1))^m ,((y^-4)*t)^n,

```

```

((y*x*y^2)*t^(y^-1))^o, ((y*x*y^2)*t)^p,
(( x * y * x *y^3 * x * y^-3)*t^(y^-1))^q,
(( x * y * x * y^3 * x * y^-3)*t)^r,
(y*t^(y^-2*x^-1))^s, (y*t)^u ,
((y^3)*t^(y^-2*x^-1))^v, ((y^3)*t)^w >;

H:=sub<G|x,y,t * y^2 * t * y^-2 * t>;
f,G1,k:=CosetAction(G,H);
#G1;
CompositionFactors(G1);

IH:=sub<G1|f(x),f(y),f(t * y^2 * t * y^-2 * t)>;
CompositionFactors(IH);

IN:=sub<G1|f(x),f(y)>;

NL:=NormalLattice(G1);
DC:=[Id(G1),f(t * y * t * y * t),f(t),f(t*y*t)];

prodim := function(pt, Q, I)

v:= pt;

for i in I do

v:=v^(Q[i]);

end for;

return v;

end function;

ts:=[Id(G1): i in [1..12]];

ts[1]:=f(t);

ts[2]:=f(t^(y^3));

```

```

ts[3]:=f(t^((y^4)*x));

ts[4]:=f(t^(y^-1));

ts[5]:=f(t^(y^-2));

ts[6]:=f(t^((y^7)*x));

ts[7]:=f(t^(y^4));

ts[8]:=f(t^(y^7));

ts[9]:=f(t^(y^2));

ts[10]:=f(t^y);

ts[11]:=f(t^(y^6));

ts[12]:=f(t^(y^5));

cst:=[null : i in [1..175]] where null is

[Integers() 1];

for i :=1 to 12 do

cst[prodim(1,ts,[i])]:=i];

end for;

m:=0;

for i in [1..175] do if cst[i] ne [] then m:=m+1;

end if; end for;m;

Orbits(N);

```

```

/* {1,2,3,4,5,6,7,8,9,10,11,12}*/

for i in [1..4] do

for g in IH do

for h in IN do

if ts[1] eq g*(DC[i])^h then i;

break;break;end if;end for;end for;end for;


N1:=Stabilizer(N,1);
N1;

#N1;

Orbits(N1);


for i in [1..4] do

for g in IH do

for h in IN do

if ts[1]*ts[1] eq g*(DC[i])^h then i;

break;break;end if;end for;end for;end for;

/*1=[*]=identity one*//*GOES BACK to [*]*/

```

```

for i in [1..4] do

for g in IH do

for h in IN do

if ts[1]*ts[2] eq g*(DC[i])^h then i;

break;break;end if;end for;end for;end for;

/*4=DC[4]=[12]*//*NEW*/

```

```

for i in [1..4] do

for g in IH do

for h in IN do

if ts[1]*ts[3] eq g*(DC[i])^h then i;

break;break;end if;end for;end for;end for;

/*3=DC[3]=[1]*//* GOES TO ITSELF*/

```

```

N12:=Stabilizer(N1,2);

```

```

SS:={ [1,2] };

```

```

SS:=SS^N;

```

```

SS;

```

```

#SS;

```

```

Seqq:=Setseq(SS);Seqq;

```

```

for i in [1..#SS] do

for n in IH do

if ts[1]*ts[2] eq

n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]

then print Rep(Seqq[i]);

end if; end for;end for;

N12;

#N12;

N12s:=N12;

for g in N do if 1^g eq 1 and 2^g eq 2 then

N12s:=sub<N|N12s,g>;end if; end for;

#N12s;

T12:=Transversal(N,N12s);

for i in [1..#T12] do

ss:=[1,2]^T12[i];

cst[prodim(1,ts,ss)]:=ss;end for;

m:=0;for i in [1..175] do if cst[i] ne []

then m:=m+1;end if;end for;m;

Orbits(N12s);

```

```

for i in [1..4] do
  for g in IH do
    for h in IN do
      if ts[1]*ts[2]*ts[1] eq g*(DC[i])^h then i;
      break;break;end if;end for;end for;end for;

```

```

for i in [1..4] do
  for g in IH do
    for h in IN do
      if ts[1]*ts[2]*ts[2] eq g*(DC[i])^h then i;
      break;break;end if;end for;end for;end for;

```

```

for i in [1..4] do
  for g in IH do
    for h in IN do
      if ts[1]*ts[2]*ts[3] eq g*(DC[i])^h then i;
      break;break;end if;end for;end for;end for;

```

```

for i in [1..4] do
  for g in IH do

```

```

for h in IN do

if ts[1]*ts[2]*ts[4] eq g*(DC[i])^h then i;

break;break;end if;end for;end for;end for;


N123:=Stabilizer(N,[1,2,3]);

SS:={[1,2,3]};

SS:=SS^N;

#SS;

Seqq:=Setseq(SS);

for i in [1..#SS] do

for n in IH do

if ts[1]*ts[2]*ts[3] eq

n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]

then print Rep(Seqq[i]);

end if; end for;end for;

N123;

#N123;


N123:=Stabiliser(N,[1,2,3]);
N123s:=N123;

for g in N do if 1^g eq 1 and 2^g eq 2 and 3^g eq 3 then

```



```

N123s:=sub<N|N123s,g>;end if; end for;

#N123s;
/* ONLY NEED TO PROVE THIS ONE*/

for g in N do if 1^g eq 2 and 2^g eq 1 and 3^g eq 8 then
N123s:=sub<N|N123s,g>;end if; end for;

for g in N do if 1^g eq 8 and 2^g eq 3 and 3^g eq 2 then
N123s:=sub<N|N123s,g>;end if; end for;#N123s;

for g in N do if 1^g eq 3 and 2^g eq 8 and 3^g eq 1 then
N123s:=sub<N|N123s,g>;end if; end for;#N123s;

T123:=Transversal(N,N123s);
for i in [1..#T123] do
ss:=[1,2,3]^T123[i];
cst[prodim(1,ts,ss)]:=ss;end for;
m:=0;for i in [1..175] do if cst[i] ne []
then m:=m+1;end if;end for;m;
Orbits(N123s);

for i in [1..4] do
for g in IH do
for h in IN do
if ts[1]*ts[2]*ts[3]*ts[3] eq g*(DC[i])^h then i;
break;break;end if;end for;end for;end for;

for i in [1..4] do
for g in IH do
for h in IN do
if ts[1]*ts[2]*ts[3]*ts[4] eq g*(DC[i])^h then i;
break;break;end if;end for;end for;end for;

```

```
/* THIS IS ISO TYPE of MAX H */  
A:=AutomorphismGroup(Alt(6));  
PA:=PermutationGroup(A);  
IsIsomorphic(PA,IH);  
  
/* so H has iso type Automorphism(Alt(6))*/  
so G1 has iso type PSU(3,5):2*/
```

Appendix J

MAGMA CODE DCE

PSL(2,11) over A_4

```

S:=Sym(6);
xx:=S!(1,4)(3,6);
yy:=S!(1,2,3)(4,5,6);
N:=sub<S|xx,yy>;
#N;
G<x,y,t>:=Group<x,y,t|x^2, y^3, (x*y)^3, t^3, (t,x^(y^-1)), t^x=t^2,
(x*t)^0, (y*t)^0, (x*y*t^y)^5,
(x*t^(x*y))^0, (t*t^y)^0>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
#G1;

CompositionFactors(G1);
#DoubleCosets(G,sub<G|x,y>, sub<G|x,y>);
DoubleCosets(G,sub<G|x,y>, sub<G|x,y>);
DC:=[ f( Id(G)), f( t * y^-1 * t * y * t^-1), f( t * y * t * y^-1 * t^-1),
f( t * y^-1 * t), f( t),f( t * y * t),
f( t * y * t * y * t), f( t * y * t * y^-1 * t)];
IN:=sub<G1|f(x),f(y)>;

```

```

ts := [ Id(G1): i in [1 .. 6] ];
ts[1]:=f(t);
ts[2]:=f(t^(y));
ts[3]:=f(t^(y^2));
/* ts[4]:=f(t^(x));
ts[5]:=f(t^(x*y));
ts[6]:=f(t^(x*y^2));*/
ts[4]:=ts[1]^2;
ts[5]:=ts[2]^2;
ts[6]:=ts[3]^2;

Index(G,sub<G1|x,y>);

cst := [null : i in [1 .. 55]] where null is [Integers() | ];
prodim := function(pt, Q, I)
v := pt;
for i in I do
v := v^(Q[i]);
end for;
return v;
end function;
for i := 1 to 6 do
cst[prodim(1, ts, [i])] := [i];
end for;
m:=0; for i in [1..55] do if cst[i] ne [] then m:=m+1; end if; end for;m;

/* FIRST DOUBLE COSET*/
Orbits(N);

for i in [1..8] do
for g,h in IN do
if ts[1] eq g*(DC[i])^h then i; break; break; end if; end for; end for;

/* SECOND DOUBLE COSET*/
N1:=Stabiliser(N,1);
Orbits(N1);
#N1;
N1;

```

```

S:={[1]};S;
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]
eq g*ts[Rep(SSS[i])[1]]
then print SSS[i];
end if; end for; end for;

N1s:=sub<N|N1>;
tr1:=Transversal(N,N1s);
for i:=1 to #tr1 do
ss:=[1]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..55] do if cst[i] ne []
then m:=m+1;
end if; end for;m;

for n in N do if 1^n eq 1 then n;N1s:=sub<N|N1s,n>;end if;end for;
N1s;
#N1s;

Orbits(N1s);

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[1] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[4] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

```

```

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[3] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

/* THRID DOUBLE COSET*/
N12:=Stabiliser(N,[1,2]);
Orbits(N12);
#N12;
N12;

S:={[1,2]};S;
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;

N12s:=sub<N|N12>;
tr1:=Transversal(N,N12s);
for i:=1 to #tr1 do
ss:=[1,2]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..55] do if cst[i] ne []
then m:=m+1;
end if; end for;m;

for n in N do if 1^n eq 1 and 2^n eq 2

```

```

then n;N12s:=sub<N|N12s,n>;end if;end for;

  N12s;
#N12s;
#N/#N12s;
Orbits(N12s);

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[1] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[2] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

/* NOTE ts[2]*ts[2] = ts[5];
> Orbits(N1s);
[
  GSet{@ 1 @},
  GSet{@ 4 @},
  GSet{@ 2, 5 @},
  GSet{@ 3, 6 @}
Nt1t2 in DC#4
and Nt1t2t2=Nt1t5 in DC#4.

remember that t's are of order 3.
So Nt1t2t2=Nt1t2^2 =Nt1t4 belongs [1 4].
Note ts[4]=ts[1]^2, ts[5]=ts[2]^2, ts[6]=ts[3]^2 */

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[3] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

for i in [1..8] do
for g,h in IN do

```

```

if ts[1]*ts[2]*ts[4] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

```

```

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[5] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

```

```

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[6] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

```

```

/* FOURTH DOUBLE COSET*/
N13:=Stabiliser(N,[1,3]);
Orbits(N13);
#N13;
N13;

```

```

S:={[1,3]};S;
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[3]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];end if; end for; end for;

```

```

N13s:=sub<N|N13>;
tr1:=Transversal(N,N13s);
for i:=1 to #tr1 do
ss:=[1,3]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..55] do if cst[i] ne []
then m:=m+1;
end if; end for;m;

```



```

for n in N do if 1^n eq 1 and 3^n eq 3 then n;
N13s:=sub<N|N13s,n>;end if;end for;

```

```

/* PROVE 13=16*/

```

```

#N/#N13s;
Orbits(N13s);

```

```

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[3]*ts[1] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

```

```

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[3]*ts[2] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

```

```

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[3]*ts[3] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

```

```

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[3]*ts[4] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

```

```

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[3]*ts[5] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[3]*ts[6] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

/* FIFTH DOUBLE COSET*/
N121:=Stabiliser(N,[1,2,1]);
Orbits(N121);
#N121;
N121;

S:={[1,2,1]};S;
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]*ts[1]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

N121s:=sub<N|N121>;
tr1:=Transversal(N,N121s);
for i:=1 to #tr1 do
ss:=[1,2,1]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..55] do if cst[i] ne []
then m:=m+1;
end if; end for;m;

for n in N do if 1^n eq 1 and 2^n eq 2 and 1^n eq 1
then n;N121s:=sub<N|N121s,n>;end if;end for;

```

```

#N/#N121s;
Orbits(N121s);

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[1]*ts[1] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[1]*ts[2] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[1]*ts[3] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[1]*ts[4] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[1]*ts[5] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[1]*ts[6] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

/* SIXTH DOUBLE COSET*/
N124:=Stabiliser(N,[1,2,4]);

```

```

Orbits(N124);
#N124;

N124;
S:={1,2,4};S;
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]*ts[4]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

N124s:=sub<N|N124>;
tr1:=Transversal(N,N124s);
for i:=1 to #tr1 do
ss:=[1,2,4]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..55] do if cst[i] ne []
then m:=m+1;
end if; end for;m;

for n in N do if 1^n eq 1 and 2^n eq 2 and 4^n eq 4
then n;N124s:=sub<N|N124s,n>;end if;end for;

for n in N do if 1^n eq 5 and 2^n eq 3 and 4^n eq 2
then n;N124s:=sub<N|N124s,n>;end if;end for;

for n in N do if 1^n eq 6 and 2^n eq 4 and 4^n eq 3
then n;N124s:=sub<N|N124s,n>;end if;end for;

#N/#N124s;

Orbits(N124s);

#N/#N124s;

Orbits(N124s);

```

```

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[4]*ts[1] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

```

```

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[4]*ts[4] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

```

```

/* SEVENTH DOUBLE COSET*/
N126:=Stabiliser(N,[1,2,6]);
Orbits(N126);
#N126;

```

```

N126;

```

```

S:={[1,2,6]};S;
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]*ts[6]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

```

```

N126s:=sub<N|N126>;
tr1:=Transversal(N,N126s);
for i:=1 to #tr1 do
ss:=[1,2,6]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;

```

```

m:=0; for i in [1..55] do if cst[i] ne []
then m:=m+1;
end if; end for;m;

for n in N do if 1^n eq 1 and 2^n eq 2 and 6^n eq 6
then n;N126s:=sub<N|N126s,n>;end if;end for;

for n in N do if 1^n eq 2 and 2^n eq 3 and 6^n eq 4
then n;N126s:=sub<N|N126s,n>;end if;end for;

for n in N do if 1^n eq 3 and 2^n eq 1 and 4^n eq 5
then n;N126s:=sub<N|N126s,n>;end if;end for;

#N/#N126s;

Orbits(N126s);

#N/#N126s;

Orbits(N126s);

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[6]*ts[1] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[6]*ts[6] eq g*(DC[i])^h then i;
break; break; end if; end for; end for;

/* EIGHTH DOUBLE COSET*/
N134:=Stabiliser(N,[1,3,4]);
Orbits(N134);
#N134;
N134;

```

```

S:={1,3,4};S;
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[3]*ts[4]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Re
p(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

N134s:=sub<N|N134>;
tr1:=Transversal(N,N134s);
for i:=1 to #tr1 do
ss:=[1,3,4]^tr1[i];
cst[prod(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..55] do if cst[i] ne []
then m:=m+1;
end if; end for;m;

for n in N do if 1^n eq 1 and 3^n eq 3 and 4^n eq 4
then n;N134s:=sub<N|N134s,n>;end if;end for;

for n in N do if 1^n eq 2 and 3^n eq 4 and 4^n eq 5
then n;N134s:=sub<N|N134s,n>;end if;end for;

for n in N do if 1^n eq 6 and 3^n eq 5 and 4^n eq 3
then n;N134s:=sub<N|N134s,n>;end if;end for;

#N/#N134s;

Orbits(N134s);

#N/#N134s;

Orbits(N134s);

```

```
for i in [1..8] do
  for g,h in IN do
    if ts[1]*ts[3]*ts[4]*ts[1] eq g*(DC[i])^h then i;
    break; break; end if; end for; end for;
```

```
for i in [1..8] do
  for g,h in IN do
    if ts[1]*ts[3]*ts[4]*ts[4] eq g*(DC[i])^h then i;
    break; break; end if; end for; end for;
```


Appendix K

MAGMA CODE DCE

PSL(2,11) over Maximal Subgroup A_5 and A_4

```

S:=Sym(6);

xx:=S!(1,4)(3,6);

yy:=S!(1,2,3)(4,5,6);

N:=sub<S|xx,yy>;

#N;

G<x,y,t>:=Group<x,y,t|x^2, y^3, (x*y)^3, t^3, (t,x^(y^-1)),
t^x=t^2, (x*y*t^y)^5>;

H:=sub<G|x,y,y * x * y^-1*t^x*t^(x*y^2)*t>;

f,G1,k:=CosetAction(G,H);

```

```

IH:=sub<G1|f(x),f(y), f(y * x * y^-1*t^x*t^(x*y^2)*t)>;

#sub<G1|f(x),f(y), f(y * x * y^-1)*ts[4]*ts[6]*ts[1]>;

#DoubleCosets(G,sub<G|x,y,y * x * y^-1*t^x*t^(x*y^2)*t>, sub<G|x,y>);

/* 3 */

IN:=sub<G1|f(x),f(y)>;

ts := [ Id(G1): i in [1 .. 6] ];

ts[1]:=f(t); ts[2]:=f(t^(y)); ts[3]:=f(t^(y^2));

ts[4]:=ts[1]^2; ts[5]:=ts[2]^2; ts[6]:=ts[3]^2;

Index(G,sub<G|x,y>);

H:=sub<G|x,y,y * x * y^-1*t^x*t^(x*y^2)*t>;

IH:=sub<G1|f(x),f(y), f(y * x * y^-1*t^x*t^(x*y^2)*t)>;

#sub<G1|f(x),f(y), f(y * x * y^-1)*ts[4]*ts[6]*ts[1]>;

/* 60 */

#sub<G|x,y,y * x * y^-1*t^x*t^(x*y^2)*t>;

/* 60 */

/*

*/

DoubleCosets(G,sub<G|x,y,y * x * y^-1*t^x*t^(x*y^2)*t>, sub<G|x,y>);

/*{ <GrpFP, Id(G), GrpFP>, <GrpFP,

```

```

DoubleCosets(G,sub<G|x,y,y * x * y^-1*t^x*t^(x*y^2)*t>, sub<G|x,y>);

{ <GrpFP, Id(G), GrpFP>, <GrpFP, t * y^-1 * t, GrpFP>, <GrpFP, t, GrpFP> }
, GrpFP>, <GrpFP, t, GrpFP> }*/

DC:= [Id(G1),f(t),f(y^-1*y*t * y^-1 * t)];

cst := [null : i in [1 .. 11]] where null is [Integers() | ];

prodim := function(pt, Q, I)
v := pt;

for i in I do
v := v^(Q[i]);
end for;
return v;
end function;
for i := 1 to 6 do
cst[prodim(1, ts, [i])] := [i];
end for;
m:=0; for i in [1..11] do if cst[i] ne [] then m:=m+1; end if; end for;m;

for i in [1..3] do
for g in IH do for h in IN do
if ts[1]*ts[1] eq g*(DC[i])^h then i;
break; break; end if; end for;end for;end for;
/*2/

for i in [1..3] do
for g in IH do for h in IN do
if ts[1]*ts[4] eq g*(DC[i])^h then i;
break; break; end if; end for;end for;end for;
/*1*/

for i in [1..3] do
for g in IH do for h in IN do

```

```

if ts[1]*ts[2] eq g*(DC[i])^h then i;
break; break; end if; end for;end for;end for;
/*3*/

```

```

for i in [1..3] do
for g in IH do for h in IN do
if ts[1]*ts[3] eq g*(DC[i])^h then i;
break; break; end if; end for;end for;end for;
/*2*/

```

```

/*Ht1t3= Ht4t6t1t1t3=H(t4t6t1)^{(1, 4)(2, 5)}t1t3

```

```

=Ht1t6t4t1t3=Ht1t6t3=Ht1

```

```

*/

```

```

/* Ht1t2t3=Ht5t4

```

```

Ht4t6t1=H, Ht4t6t1t1t3=Ht1t3

```

```

Ht1t3=Ht4t6t1t1t3=Ht4t6t4t3=Ht3t1t2t4t3=Ht3t1t2 (1, 5, 6)(2, 3, 4)t6t1t5

```

```

=Ht4t5t3t6t1t5=Ht4t5t1t5

```

```

(1, 6, 5)(2, 4, 3)

```

```

[ 4, 3 ]

```

```

[ 6, 1, 5 ]

```

```

(1, 6, 5)(2, 4, 3)

```

```

[ 4, 3 ]

```

```

[ 6, 1, 5 ]

```

```

(1, 3, 2)(4, 6, 5)

```

```

[ 4, 6 ]

```

```

[ 3, 1, 2 ]

Ht4=Ht4t6

/* (1, 3, 2)(4, 6, 5)

[ 5, 4 ]

[ 1, 2, 3 ]

Ht1t2t3=Ht5t4t2t5=Ht5,  Ht4t6t1(1, 2, 3)(4, 5, 6)= Ht4t6t1(1, 2, 3)(4, 5, 6)

= H(1, 2, 3)(4, 5, 6)(1, 2, 3)(4, 5, 6)^{-1}t4t6t1(1, 2, 3)(4, 5, 6)

= H(t4t6t1)^{(1, 2, 3)(4, 5, 6)}=Ht5t4t2*/

S:={[1,2]};

SS:=S^N;

SSS:=Setseq(SS);

for i in [1..#SSS] do

for g in IH do if ts[1]*ts[2]

eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]

then print SSS[i];

end if; end for; end for;

/*{

[ 1, 2 ]

}

```

```

{
    [ 5, 6 ]
}

{
    [ 3, 4 ]
}

*/

N12:=Stabiliser(N,[1,2]);

#N12;

N12s:=N12;

for n in N do if [1,2]^n eq [5,6] then N12s:=sub<N|N12s,n>; end if; end for;

[1,2]^N12s;

/* {@
    [ 1, 2 ],
    [ 5, 6 ],
    [ 3, 4 ]
@}

*/

#N12s;

/*3*/

```

```

tr1:=Transversal(N,N12s);

for i:=1 to #tr1 do

ss:=[1,2]^tr1[i];

cst[prodim(1,ts,ss)]:=ss;

end for;

m:=0; for i in [1..11] do if cst[i] ne []

then m:=m+1;

end if; end for;m;

Orbits(N12s);

/* [

    GSet{@ 1, 5, 3 @},

    GSet{@ 2, 6, 4 @}

] */

/*Ht1t2t5=Ht1 in [1] */

/*Ht1t2t2=Ht1t5= H(t1t2)^(2, 5)(3, 6) in [1,2]*/

```

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